The projective geometry of the spacetime yielded by relativistic positioning systems and relativistic location systems

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As well accepted now, current positioning systems such as GPS, Galileo, Beidou, etc. are not primary, relativistic systems. Nevertheless, genuine, relativistic and primary positioning systems have been proposed recently by Bahder, Coll *et al.* and Rovelli to remedy such prior defects. These new designs all have in common an equivariant conformal geometry featuring, as the most basic ingredient, the spacetime geometry. In a first step, we show how this conformal aspect can be the four-dimensional projective part of a larger five-dimensional geometry. Our aim has been then to explore and collect all of the geometric, physical consequences of this projective geometry in such spacetime context and ask for a physical process, the implementation of which could reveal this fifth dimension. We find that the latter is physically obtained from a fifth time stamp effectively yielded from a new localization protocol that we present jointly with this projective geometry. Based on this fifth supplementary parameter, beside the four usual ones identified with the four basic time stamps broadcast by any positioning system, we deduce the four-dimensional projective geometry governing spacetime. The former is completely detailed, *i.e.*, the projective Cartan connection and its projective curvature are computed. As a result, the Einstein tensor appears to be algebraically remarkable in this projective context. In particular, this leads to a new surprisingly result in the complete integrability of the Einstein equations while the well-known incompleteness of their usual non-projective version is unquestioned. Also, it may validate completely or gives naturally true physical grounds to the old model due Yano, Ohgane and Ishihara or the new approach developed by Bradonjić and Stachel similar to the one due to Schouten, Haantjes and van Dantzig unifying electromagnetism and gravitation in which the homogeneous coordinates remained till now unidentified physically.

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I. INTRODUCTION - THE COLL-FERRANDO-MORALES-TARENTOLA PROTOCOLS

Historically, the most "basic" first tool for positioning events in Minkowski spactime was based on the so-called *Marzke-Wheeler protocol* [MW64] (see Appendix A). Then, eight years later, Ehlers, Pirani and Schild [EPS72] improved the latter protocol in the framework of the general relativity, and they defined a metric g with a Lorentz signature on a given spacetime manifold $\mathcal{M}$ starting from what we call a '*potential of metric* v' or a 'distance function v' on $\mathcal{M}$. The metric g obtained in this way is uniquely defined at any event $e \in \mathcal{M}$, up to a conformal factor, from any parameterized time-like worldline W such that $e \in W$ and $t(e) = 0$, where t is any time parameterization given on W (see Figure I.1). One of the essential ingredients in this

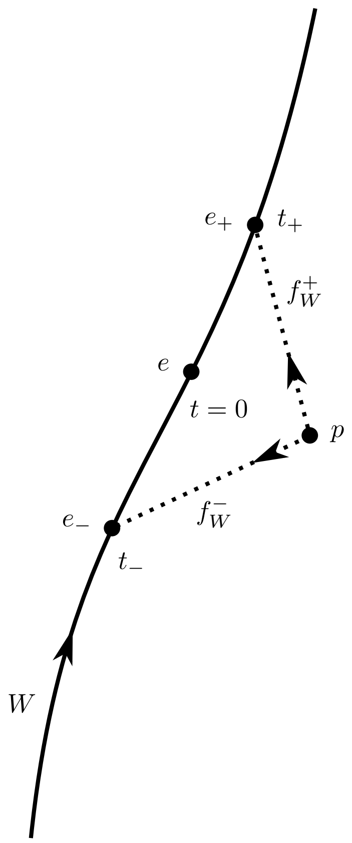


Figure I.1. *The Ehlers-Pirani-Schild protocol.*

definition are the so-called '*radar coordinates*' r_+ and r_- defined themselves respectively from the so-called '*message functions*' f^+_W and f^-_W. The latter are maps associating any event $p \in \mathcal{M}$ with two events $e_\pm \in W$ such that e_+ (resp. e_-) is on the future (resp. past) light cone of p. Then, the radar coordinates are maps from $\mathcal{M}$ to $\mathbb{R}$ such that $r_\pm \equiv t \circ f^\pm_W$ and the potential of metric v is defined by the product of r_+ and r_-:

$$v \equiv -\, r_+ \, r_- \, . \tag{1.1}$$

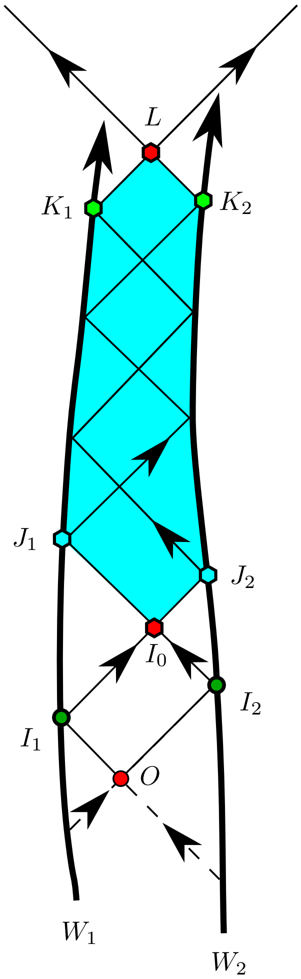
Then, the metric g is obtained from the Hessian of v or, equivalently, from the following formula:

$$g \equiv -\, dr_- \otimes dr_+ - dr_+ \otimes dr_- \equiv -2\, dr_+ \odot dr_- \, , \tag{1.2}$$

where d is the exterior derivative defined on $\mathcal{M}$ and $\odot$ represents the symmetrized tensor product. At the event $e \in W$, the metric g is defined up to conformal factors due to any change of parameterization t of W satisfying the constraint $t(e) = 0$.

Unfortunately, this protocol cannot be really implemented physically because the value $t_+ = r_+(p)$ of the radar coordinate r_+ is obtained by any observer at p from signals coming from the future. To circumvent this difficulty, Coll, Ferrando, Morales and Tarentola proposed an alternative protocol in four-dimensional spacetimes [Col85, Col01a, Col02, CT03] (see also Blagojević *et al.* [BGHO02], Bahder [Bah01, Bah03, Bah04] and Rovelli [Rov02a, Rov02b]), but also, in particular, primarily in two-dimensional spacetimes [CFM06b, CFM06a, CP06, CFM10a, CFM10b] with at least two worldlines W_1 and W_2 (see Figure I.2).

With their protocol, only signals coming from the past are utilized to deduce the metric g or the spacetime positions of users in what we call a '*hexagonal domain*' $I_0 J_1 K_1 L K_2 J_2 I_0$ of the spacetime between the two given worldlines W_1 and W_2. Their protocol can be presented in a simplified version in a two-dimensional spacetime as follows. Let us consider the event O corresponding to an ignition event from which two flashes of light are emitted toward the two time-like worldlines W_1 and W_2. These two flashes are received on W_1 and W_2 at the events I_1 and I_2 respectively and they ignite or set to zero the two parameterizations t_1 and t_2 given on these two worldlines. Then, the point I_0 has the coordinates ($t_1 = 0$, $t_2 = 0$) (see Fig. I.2). Moreover, we consider that the two worldlines are the trajectories of two emitters which send



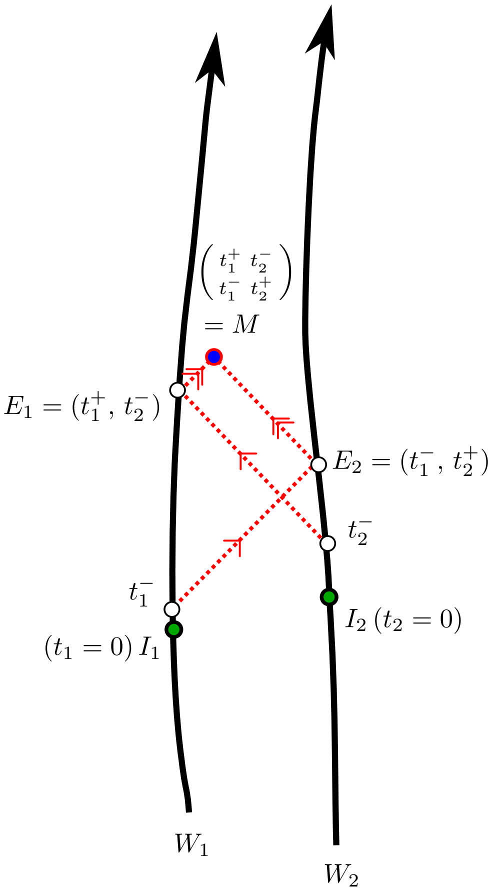


Figure I.2. Figure on the left: the "hexagonal" domain $I_0J_1K_1LK_2J_2I_0$. Figure on the right: the dashed lines are the light-like paths of the signals carrying the time stamps with values $t_1^\pm$ and $t_2^\pm$ by successive echoes. For instance, at the event E_2, we have an echo with the reception of the time stamp with value t_1^- and a sending to M of a pair of time stamps with values $(t_1^-,\, t_2^+)$.

themselves — via light-like paths — time stamps and broadcast continuously those 'time stamps' they receive by producing, somehow, echoes.

Then, it can be shown that at any event M in the hexagonal domain, four numbers $(t_1^+, t_2^+, t_1^-, t_2^-)$ carried by the time stamps can be obtained and utilized by any user at M to build a grid or a chart in the hexagonal domain and to deduce the spacetime positions of M and the two emitters, say E_1 and E_2 (on W_1 and W_2), in this grid. In the latter, the event M

has the coordinates t_1^+ and t_2^+, *i.e.*, $M \equiv (t_1^+, t_2^+)$, whereas the two emitters have the emission coordinates $E_1 \equiv (t_1^+, t_2^-)$ and $E_2 \equiv (t_1^-, t_2^+)$. Moreover, the spatial distance between the two events of ignition I_1 and I_2 is not defined, but can be posed as a standard of distance associated with the relativistic positioning system defined by the two worldlines W_1 and W_2. What is really of importance is to be able in this situation to define the protocol putting in correspondence the standards associated with two such different positioning systems. Then, by standard matching we may tend possibly toward a limit case with a positioning system composed of two intersecting worldlines for which a standard is not necessary.[1] But, as in the Marzke-Wheeler protocol, we can also define a potential of metric v. Indeed, we can take v as the Lorentzian distance function $d(O,M)$ between O and M such that

$$v \equiv d(O,M) \equiv -\,t_1(f^-_{W_1}(M))\,t_2(f^-_{W_2}(M)) = -\,t_1^+\,t_2^+\,. \tag{1.3}$$

Then, the metric g is defined by the relation:

$$g \equiv -2\,dt_1^+ \odot dt_2^+\,. \tag{1.4}$$

In a four-dimensional spacetime, the situation is quite similar with four emitters rather than with two emitters only. But, different approaches can be taken a priori to describe the possible underlying geometry of the spacetime deduced from such protocols. One of them, generalizing the Coll-Ferrando-Morales-Tarentola protocol (CFMT), was investigated by Ferrando and Sáez in the framework of the '$2+2$ *warped spacetimes*' by duplicating the two-dimensional approach [FS10]. In the present paper, we determine the spacetime geometry deduced from the strict general four-dimensional protocol such as the one described for instance in the so-called SYPOR relativistic positioning system [CT03]. The main goal is, at first, to define a metric g in a four-dimensional spacetime generalizing the formula (1.4) and then, to deduce the spacetime geometry.

[1] The importance of such an intersection point —the so-called '*cut event*'— of the two worldlines of the two emitters has been clearly pointed out by B. Coll, J. J. Ferrando and J. A. Morales [CFM06b, see footnote 11].

II. THE METRIC AND THE "ENERGYSPACETIME" – A FUNDAMENTAL EXEMPLE

A. A fundamental example

Any Lorentzian metric g can be always put univocally in the following general form in a $\{\ell\ell\ell\ell\}$-coframe (ℓ for light):

$$g = -\sum_{i<j=1}^{4} \nu_i \nu_j \, d\tau_i \odot d\tau_j \, , \tag{2.1}$$

where the ν's are positive (nonvanishing) functions depending on four time coordinates τ_k which are the time parameterizations (stamps) of four worldlines travelled by four emitters (see also analogous situations in [KM07, KMW10, KM10, KM12]).

More precisely, we proved the two following theorems (see Appendix B for the proofs) we call the "*factorization theorems*." Let π_n be the trivial fibration $\pi_n : \mathcal{M}^2 \equiv \mathcal{M}\times\mathcal{M} \longrightarrow \mathcal{M}$, corresponding to the projection onto the first factor and where $\dim \mathcal{M} = n$ in full generality. We denote by $J_k(\pi_n)$ the fiber bundle of jets of order $k \geqslant 0$ of the local smooth sections of π_n. In particular, we have $J_0(\pi_n) \equiv \mathcal{M}^2$ with local coordinates $(\tau, \psi) \equiv (\tau^1, \ldots, \tau^n, \psi^1, \ldots, \psi^n)$ where τ is the *source* and ψ is the *target*. Furthermore, let

$$\psi_1 \equiv (\tau^1, \ldots, \tau^n, \psi^1, \ldots, \psi^n, \psi^1_1, \psi^1_2, \ldots, \psi^i_j, \ldots, \psi^n_{n-1}, \psi^n_n)$$

be a local system of coordinates on $J_1(\pi_n)$. We denote also by $\Pi_k(\pi_n) \subset J_k(\pi_n)$ the set of invertible elements of $J_k(\pi_n)$, *i.e.*, the set of k-jets of local smooth diffeomorphisms on $\mathcal{M}$. $\Pi_k(\pi_n)$ is a groupoid with source map $\alpha_k : \Pi_k(\pi_n) \longrightarrow \mathcal{M}$ where $\mathcal{M}$ is the first factor of $\mathcal{M}^2$ and the target map $\beta_k : \Pi_k(\pi_n) \longrightarrow \mathcal{M}$ where we project onto the second factor. Also, we denote by $\mathit{\Pi}_k(\pi_n)$ the presheaf of germs of local smooth α_k-sections of $\Pi_k(\pi_n)$. Then, we consider any solution of the system of PDEs (2.3) below as a sub-manifold of $\Pi_1(\pi_n)$ transversal to the α_k-fibers and defined from the following system $\mathcal{R}_1$ of equations on the presheaf $\mathit{\Pi}_1(\pi_n)$:

$$\mathcal{R}_1 : \quad \sum_{r,s=1}^{n} g_{rs}(\psi)\, \psi^r_i\, \psi^s_j - \epsilon_{ij}\, \nu_i(\tau)\, \nu_j(\tau) = 0 \, , \qquad i, j = 1, \ldots, n \, , \tag{2.2}$$

where $\nu_i(\tau) > 0$. Then, we have the following first theorem.

Theorem 1. (First factorization theorem). *If $n \leqslant 4$, there always exists a smooth local diffeomorphism $f(\tau) = \psi$ and n smooth positive functions $\nu_i(\tau)$, both defined on an open neighborhood $V \subset U$ of any given point of U, such that for all $\tau \equiv (\tau^1, \dots, \tau^n) \in V$ the relations*

$$\tilde{g}_{ij} \equiv \sum_{r,\,s=1}^{n} g_{rs}(f)(\partial_i f^r)(\partial_j f^s) = \epsilon_{ij}\,\nu_i\,\nu_j\,, \qquad i,j = 1,\dots,n\,, \tag{2.3}$$

hold with $\epsilon_{ij} = \mathrm{sgn}(g_{ij}) = \mathrm{sgn}(\tilde{g}_{ij})$ whenever $i \neq j$ and $\epsilon_{ij} = 0$ otherwise. Then, we say that the "generic" metric $\tilde{g}$ is ℓ-equivalent *to g (through f).*

And also, if $\mathcal{M}$ is *time oriented*, *i.e.*, there exists a complete (future time-like) vector field ξ on $\mathcal{M}$, then, we have the following second theorem:

Theorem 2. (Second factorization theorem). *If $n = 4$, then, given a Lorentzian metric g on $\mathcal{M}$ assumed to be time oriented, connected and simply connected, then, there exists only one smooth diffeomorphism $f^i(\tau) \equiv \psi^i$ being a solution of $\mathcal{R}_1$ of which the Jacobian matrix is an element of $O(4,\mathbb{R})$; and, as a result, there is a unique set of four positive functions ν_i. Also, the unique ℓ-equivalent metric field $\tilde{g}$ is* 'isometrically equivalent' *to g. Then $\tilde{g}$ is said to be* ℓ-isometric *to g and* ℓ-generic.

It must be noted that not any given Lorentzian metric g can be diagonalized without implicit assumptions if $\dim \mathcal{M} \geqslant 4$. Indeed, we have always an (pseudo-)orthogonal coframe only in a manifold of dimension n less than or equal to 3. We recall that a pseudo-orthogonal coframe is a cobasis of 1-forms σ_i ($i = 1, \dots, n$) such that $g \equiv \sum_i^n \epsilon_i\,\sigma_i \odot \sigma_i$ and $\sigma_i \wedge d\sigma_i = 0$ with $\epsilon_i = \pm 1$. The vanishing of the Weyl tensor is, for instance, a sufficient condition in a torsion-free Riemann structure (*i.e.*, the connection on $\mathcal{M}$ is the Levi-civita connection) to have such pseudo-orthogonal coframe in dimension $n \geqslant 4$, and then, to diagonalize g.[2]

[2] Actually, in a given coframe, the necessary and sufficient condition for this coframe to be (pseudo-)orthogonal is the vanishing of certain coefficients of the Riemann tensor R. More precisely, the coefficients $R_{ij,k\ell}$ of the Riemann tensor must vanish whenever all of the indices i, j, k and ℓ are distinct (a condition always satisfied in dimension less than or equal to 3). This is equivalent to the vanishing of all of the corresponding covariant components of the Weyl curvature tensor (obviously, a vanishing Weyl tensor involves such relations on the Riemann coefficients) [Wei72, see pp. 145–146][BCG+91, pp. 88–91][Bry99, §7 pp. 46–47]. The proof in [Bry99, §7 pp. 46–47] is given for orthogonal coordinate charts and Euclidean metrics on $\mathbb{R}$. However, because the proof is analytical, it remains valid on $\mathbb{C}$. We must just consider an Euclidean metric $g = \sum_{j=1}^4 \alpha_j \otimes \alpha_j$ on $\mathbb{C}$ with $\alpha_k \equiv i\sqrt{|a_{kk}|}\,du_k$ for three of the four indices j and, additionally, keeping the same dimensions of the various varieties or manifolds but on $\mathbb{C}$. See also, for instance, the conclusion P.15 § 4.2 "The Lorentzian case" in [GV09].

It follows, on the contrary, that a nonvanishing Weyl tensor can be compatible with a non-diagonalization whatever are 1-forms σ_i satisfying $\sigma_i \wedge d\sigma_i = 0$,[3] and then, in full generality and without any indication a priori on this vanishing, we cannot recast our geometrical approach starting with a diagonal metric g.[4] In particular, applying the well-known Newman-Penrose formalism of "null" tetrads [NP04] we must be aware that the orthogonality condition imposed by Newman and Penrose to their "null" tetrads cannot be satisfied to any set of four null vectors.

Moreover, g does not derive, in full generality, from a potential of metric ψ. But, if g admits such potential we should have the following system $\mathcal{R}_2$ of partial differential equations (assuming that the roman indices i, j, k, ... are equal to 1, 2, 3 or 4):

$$\mathcal{R}_2 \equiv \begin{cases} \partial_{ij}\psi = \nu_i \nu_j \quad \text{if } i \neq j, \\ \partial_{kk}\psi = 0, \end{cases} \tag{2.4}$$

where ∂_k represents the partial derivative with respect to the time coordinate τ_k. The *symbol* M_2 of this system of partial differential equations of degree 2 is vanishing, *i.e.*, the system $\mathcal{R}_2$ is *involutive*. Therefore, all the '*compatibility conditions*' ($\mathcal{CC}$) derive from those obtained from the first prolongation of $\mathcal{R}_2$. More precisely, we obtain:

$$\mathcal{CC} \equiv \begin{cases} \mathcal{CC}1: \quad \partial_i(\nu_i \nu_j) = 0 & \text{if } i \neq j, \\ \mathcal{CC}2: \quad \partial_i(\nu_j \nu_k) = \partial_j(\nu_i \nu_k) & \text{if } i \neq j \neq k. \end{cases} \tag{2.5}$$

From $\mathcal{CC}1$, we deduce easily that there exist differentiable functions A_{ij} depending on the time coordinates τ_k such that $\nu_i \nu_j = A_{ij}(\tau_k, \tau_h)$, where $i, j \neq k, h$, $i \neq j$ and $k \neq h$. Then, the conditions $\mathcal{CC}2$ involve that $A_{ij}(\tau_k, \tau_h) = (\partial_{ij}\psi)(\tau_k, \tau_h)$. Hence, necessarily, ψ must be a polynomial of degree at most 2 with respect to any pair of time coordinates τ_i. In addition, its

[3] We can always locally diagonalize on an open neighborhood but with a basis change matrix applied on the σ_i's not necessarily attributed to a local change of coordinates. Find basis change matrices which are also Jacobian matrices of change of coordinates maps is obviously typically the problem of the equivalence between Riemannian manifolds.

[4] Moreover, the condition $\sigma_i \wedge d\sigma_i = 0$ is verified if the Riemann scalar curvature is constant (because then, the Weyl tensor vanishes and the conditions given in footnote 2 are satisfied), and then, the σ's correspond to differentials of geodesic (see just below) emission coordinates. In addition, the σ's become also soldering forms of which the constants of structure obtained from the $d\sigma$'s are the components of the Euclidean connection given on $\mathcal{M}$ [Car08, see condition (5)][Car22a][DY84, if $n = 3$]. Indeed, let $\{\alpha_1, \ldots, \alpha_n\}$ be a coframe such that $d\alpha_k + \frac{1}{2}\sum_{i,j}^n C_k^{i,j} \alpha_i \wedge \alpha_j = 0$ (Frobenius theorem for a completely integrable Pfaff system). Also, we denote by ∇^* the dual covariant derivative defined from a Euclidean connection ω, and by $\mathcal{O}_\mathcal{M}$ the presheaf of germs of local smooth functions defined on $\mathcal{M}$. Then, we obtain with $\sigma \equiv u^i \alpha_i$ and $u^k \in \mathcal{O}_\mathcal{M}$ that $\nabla^*\sigma = \sum_{k=1}^n du^k \otimes \alpha_k - \sum_{i,k=1}^n u^k \omega_k^i \otimes \alpha_i = d\sigma - \sum_{j,k=1}^n u^k(d\alpha_k + \omega_k^j \otimes \alpha_j)$ with $\nabla_\zeta^*\sigma \equiv i_\zeta(\nabla^*\sigma)$ and where i_ζ is the interior product with respect to any vector field $\zeta \in \chi(\mathcal{M})$. Therefore, $\sigma \wedge \nabla^*\sigma = \sigma \wedge d\sigma + \sum_{j,k,h=1}^n u^k u^h (d\alpha_k + \omega_k^j \otimes \alpha_j) \wedge \alpha_h$ and $\sigma \wedge \nabla^*\sigma = \sigma \wedge d\sigma$ if $i_\zeta d\alpha_k + \omega_k^j(\zeta)\,\alpha_j \equiv 0$ for any ζ. Hence, the problem [Car08, Car22a] reduces to an equivalence problem to know if there exists a coframe $\{\tilde\alpha_1, \ldots, \tilde\alpha_n\}$ linearly defined from $\{\alpha_1, \ldots, \alpha_n\}$ with functions of structure such that $\widetilde{C}_k^{i,j}\, \tilde\alpha_i \equiv \omega_k^j$ to have $\sigma \wedge \nabla^*\sigma = 0$ and $d\tilde\alpha_k + 1/2\sum_{i,j=1}^n \widetilde{C}_k^{i,j} \tilde\alpha_i \wedge \tilde\alpha_j = 0$ satisfied. And then, considering that $\sigma \equiv \tilde u^i \tilde\alpha_i$, we have $\sigma \wedge \nabla^*\sigma = \sigma \wedge d\sigma$ and therefore $\sigma \wedge d\sigma \equiv 0$ whenever $\nabla^*\sigma \equiv \rho \wedge \sigma$, *i.e.*, σ is the dual of a geodesic vector. Then, from $\sigma_k \wedge d\sigma_k \equiv 0$, it is easy to prove (see footnote 46, p. 187) that $\sigma_k \equiv dx_k$, and thus the σ's are exact and expressed in a system of local geodesic coordinates $\{x_1, \ldots, x_n\}$.

coefficients must be also polynomials of degrees at most 2 depending only on the two remaining time coordinates. Moreover, because $\partial_{kk}\psi = 0$, then we deduce that the general solution ψ is of the form: $c\,\tau_1\tau_2\tau_3\tau_4$, where c is a real nonvanishing constant. From $\mathcal{R}_2$, we obtain that

$$\nu_i^2 = \frac{\psi}{\tau_i^2}\,. \tag{2.6}$$

Then, we must have $\psi > 0$. Therefore, we set

$$\psi \equiv c\,|\tau_1\tau_2\tau_3\tau_4| > 0\,. \tag{2.7}$$

We obtain what is called a '*fourth root metric*' and ψ can be identified with a Lorentzian distance function on the affine spacetime $\mathcal{M}$. Beside, we have a potential of metric ψ if and only if no time coordinate vanishes. It matters to notice that each 1-form σ_i such that

$$\sigma_i \equiv \nu_i\, d\tau_i \tag{2.8}$$

defines a one-dimensional involutive Pfaff system. Indeed, we have the relation of involution:

$$d\sigma_i = \frac{1}{2}\, d(\ln\psi)\wedge\sigma_i = \frac{1}{2\sqrt{\psi}}\left(\textstyle\sum_{k=1}^4 \sigma_k\right)\wedge\sigma_i\,. \tag{2.9}$$

Therefore, if we define the 1-forms λ_i such that $\lambda_i \equiv \psi^{-\frac{1}{2}}\,\sigma_i$, then we obtain exact 1-forms, *i.e.*, $d\lambda_i = 0$. Thus, the metric g can be written in the following form:

$$g = -\,\psi \sum_{i<j=1}^{4} \lambda_i \odot \lambda_j\,, \tag{2.10}$$

and there exist functions μ_i such that $d\mu_i = \lambda_i$. We deduce easily that $\mu_i = \ln|\tau_i|$,[5] and then

$$g = -\,\psi \sum_{i<j=1}^{4} d\mu_i \odot d\mu_j = -c\, e^{\phi(\mu)} \sum_{i<j=1}^{4} d\mu_i \odot d\mu_j\,, \tag{2.11}$$

where $\phi(\mu) \equiv \sum_{k=1}^4 \mu_k$. Hence, we just obtain a reparameterization of the worldline of each emitter with new time coordinates μ_i which are no more no less than the so-called '*isothermal coordinates*.' In particular, if we compute de Riemann scalar curvature $\mathcal{S}$ and the Weyl scalar curvature $\mathcal{W}$ defined from g, we obtain:

$$\mathcal{S} = \frac{4}{c}\, e^{-\phi(\mu)}\,, \qquad\qquad \mathcal{W} = 0\,. \tag{2.12}$$

[5] The absolute value indicates, nevertheless, that we loose the time orientation on the worldlines.

Thus, the spacetime manifold is conformally flat and flat at infinite times. As a consequence, the coordinates μ_k parameterize null geodesics defined by the Levi-Civita connection ∇ of g; a fact which can be checked also from the following relations:

$$\nabla_{\xi^i}\xi^i = 0\,, \qquad \xi^i \equiv e^{-\mu_i}\,\frac{\partial}{\partial\mu_i}\,, \qquad g(\xi^i,\xi^i) = 0\,. \tag{2.13}$$

In particular, we must mention that the emitters are not necessarily free falling, *i.e.*, whatever are their worldlines the conformal flatness is only due to the metric g chosen from pure parameterization considerations, independently of, for instance, the mass content in the spacetime $\mathcal{M}$ or the emitters kinematics. Hence, the spacetime is represented by a (pseudo-)Riemannian manifold which is fully disconnected from the physics but, nevertheless, perfectly suitable for the design of a relativistic positioning system since to each event is associated a set of four time stamps. This manifold would be somehow "blind" or "insensitive" to the linkage between the spacetime geometry and the spacetime physics.

This example is quite important because it indicates that the emitters must be linked in a particular way to the physics of some spacetime contents and not only embedded geometrically in a neutral way. Moreover, it involves also that physical supplementary parameters must be included in the geometrical description of the spacetime. Besides, a supplementary parameter has been shown to be necessary to avoid some strong inconsistencies in the definitions available for the notion of proper time of a time-like observer [Rub10]. It can be simply associated with a dimension of energy added beside those of space and time. We can add that in a 1+1 spacetime the CFMT protocol exhibits conformal parameters depending on accelerations (or energies) of the emitters, *i.e.*, the so-called *emitter shift parameters* [CFM10a]. These parameters are not fields defined on this two-dimensional spacetime, *i.e.*, they are associated with a third path-dependent parameter not only dependent on the spacetime position at which it is evaluated but the spacetime content, *i.e.*, the emitters and their trajectories. As a result, we obtain implicitly a 1+1+1 dimensional "*energyspacetime*."

Thus, we can conclude that a generalization in dimension 4 of a metric g defined from a potential of metric ψ compatible with a generalized CFMT protocol in a no-warped spacetime might be physically too restrictive, although well suited for a relativistic positioning system.

Futhermore, from the definition (2.1) of g, we obtain that

$$\det g = -\frac{3}{16}\,(\nu_1\,\nu_2\,\nu_3\,\nu_4)^2\,, \tag{2.14}$$

and the determinant of g must be assumed to be not constant in full generality according, for instance, to the results of Ehlers, Pirani and Schild on the conformal structures of spacetimes. Besides, we deduce also that the signature of g is -2 or, equivalently, $(+,-,-,-)$.

As a consequence, we should have a '*conformal structure*' on $\mathcal{M}$ and the values of the conformal factors would be only due to the dynamics of the relativistic positioning systems when they are explicitly in relationships with certain spacetime contents. From the results obtained by Coll *et al.*, the latter cannot be the set of free falling particles of which the physics defines the so-called (three-dimensional) '*projective structure*' on $\mathcal{M}$ [EPS72], and then, the linkage between the physics and the geometry would be obtained from the values of the conformal factors of g emerging from any given relativistic positioning system via generalized parameters (like the emitter shift parameters).

Actually, what the CFMT protocols and the causal axiomatics on conformal structures reveal, is a true path dependency via the emitter shift parameters depending explicitly on integration paths from which, if the emitters are not on geodesics, the metric field is defined. As a strong consequence, we have a so-called '*second clock effect*', *i.e.*, these relativistic positioning systems reveal surprisingly a '*Weyl structure*' coming from the satellites constellation, *i.e.*, from the relativistic positioning system, though not necessary from the spacetime manifold $\mathcal{M}$ itself (of dimension 1+1 in the CFMT protocol) [EPS72]. This path dependence is not only occurring in the context of such protocols, but also, for instance, implicitly in the fundamental historical Kundt-Hoffmann protocol [KH62] for determining the scale factor of the metric field.

In other words, *the [charts] (only) of the atlas on $\mathcal{M}$ yielded by a relativistic positioning system (RPS) might be possibly path-dependent, i.e., they might depend on the paths (worldlines) of the satellites and not only on the events at which they stand.*[6]

[6] D. Pandres, Jr. and E. L. Green described a sort of non-commutative geometry based on the non-commutativity of the partial derivatives on a space of path-dependent functionals on the spacetime with certain conformal aspects [Pan81, Pan84, Pan95, PJG03, Gre09]. Moreover, the derivatives of the charts are diffeomorphisms on the tangent spaces. It sounds like the situation encountered in the Carathéodory thermodynamics, *i.e.*, with so-called '*flags*' of 1-forms satisfying not the Frobenius conditions [BCG+91, Kum99]. Solving a flag provides path-dependent solutions if '*Monge parameterizations*' are not used. On the contrary, it means that we avoid path-dependency when using Monge parameterizations. The latter involve to consider additional parameters — the Monge parameters — increasing the dimensionality of the spacetime to another manifold with more than four dimensions. And then, in that '*Monge space*' we have only functions rather than functionals. Actually, we obtain in the present context '*multi-flags*' [KR02].

To describe the spacetime $\mathcal{M}$ via RPS, we can consider its *causal space* structure and, in particular, its set of Alexandrov chronological future opens $\mathcal{I}_o^+ \equiv \{e \in \mathcal{M} \,:\, o \ll e\} \subset \mathcal{M}$, where $\ll$, or $\gg$, is the *chronological order* on $\mathcal{M}$ [KP67, Pen72, GPS05] and where the '*basepoint*' o is a particular event (*e.g.*, the 'cut event' for instance [CFM06b, see footnote 11]). In addition, we consider also the sets $\mathcal{P}_o^+$ of *future-directed time-like* paths $\gamma_{o,e} \subset \{o\} \cup \mathcal{I}_o^+$ with the fixed basepoint o and endpoints $e \gg o$.

Furthermore, because the path-dependency involves to incorporate values of certain integrals of accelerations or forces along paths, *i.e.*, somehow, an "energy" 1-form $\alpha \in T^*\mathcal{M}$ of class C^0, we have what we call an '*energyspacetime $E\mathcal{M}^+$*' modeled over $(\mathcal{M},\, \alpha)$, the definition of which can be suggested succinctly as follows. To each chronological open $\mathcal{I}_o^+$, we associate a subset $\mathcal{E}\mathcal{I}_o^+ \subseteq \mathcal{I}_o^+ \times \mathbb{R}$ and a map $\Gamma_o : \mathcal{E}\mathcal{I}_o^+ \longrightarrow \mathcal{I}_o^+$ such that for all $(e, \epsilon) \in \mathcal{E}\mathcal{I}_o^+$ we have $\Gamma_o(e, \epsilon) = e$ if and only if there exists a path $\gamma_{o,e} \in \mathcal{P}_o^+$ such that $\epsilon = \int_{\gamma_{o,e}} \alpha$. Note that Γ_o is surjective and differentiable because α is, inter alia, of class C^0 on $\mathcal{M}$. Then, we define $E\mathcal{M}^+ \subset \mathcal{M} \times \mathbb{R}$ to be the space such that $E\mathcal{M}^+ \equiv \cup_{o \in \mathcal{M}} \mathcal{E}\mathcal{I}_o^+$, and whenever $\mathcal{M}$ is time-oriented we obtain:

Theorem 3. *If $\{o\} \cup \mathcal{I}_o^+$ is path connected and 't-complete,' then $\mathcal{E}\mathcal{I}_o^+$ is a fiber bundle over $\mathcal{I}_o^+$. Moreover, either $\mathcal{E}\mathcal{I}_o^+$ or each fiber $\Gamma_o^{-1}(e)$ where $e \in \mathcal{I}_o^+$ is compact. (See proof with the definition of 't-completness' in Appendix H, p. 229.)*

Let us note that though the 1-form α is of class C^0 on the closure $\overline{\mathcal{I}}_o^+$ of any future chronological open $\mathcal{I}_o^+$, its integral $\int \alpha$ along an achronal (non-timelike) path $\gamma_{o,e}$ can possibly diverge. This divergence can be due, in particular, to points of the $\gamma_{o,e}$'s at which the tangent vectors are light-like. In other words, we could say also that $\overline{\mathcal{I}}_o^+$ is not necessarily t-complete. Moreover, we obtain the following:

Lemma 1. *The 'energyspacetime' $E\mathcal{M}^+ \equiv \cup_{o \in \mathcal{M}} \mathcal{E}\mathcal{I}_o^+$ is a fiber bundle over $\mathcal{M}^+ \equiv \cup_{o \in \mathcal{M}} \mathcal{I}_o^+$.*

Hence, the non-local character of $E\mathcal{M}^+$ defined by the paths $\gamma_{o,e}$ is not true any longer. Then, we can assume that a point in $E\mathcal{M}^+$ is framed by five time stamps. Also, we deduce that atlases of $E\mathcal{M}^+$ with charts to $\mathbb{R}^5$ exist such that covering paths $\hat{\gamma}$ in $E\mathcal{M}^+$ can be obtained from given paths $\gamma_{o,e}$ on $\mathcal{M}$. Moreover, from the 1-form α on $\mathcal{M}^+$ we can easily define a transversal[7] 1-form Φ on $E\mathcal{M}^+$ such that $\Phi \equiv d\epsilon \equiv \alpha\,(\neq \Gamma_o^*(\alpha))$ in $T\mathcal{E}\mathcal{I}_o^+ \subset TE\mathcal{M}^+$, and then, we deduce another property from a Ehresmann's theorem (see Ehresmann's theorem on connections in Appendix G, p. 228, and [Ehr47a, Proposition 3, p.1612]) given any transversal 1-form Φ on $E\mathcal{M}^+$ over $\mathcal{M}^+$ and if the fibers $\Gamma_o^{-1}(e)$ are compact: to any path from e to e' in $\mathcal{M}^+$ corresponds univocally a well-determined homeomorphism from $\Gamma_o^{-1}(e)$ to $\Gamma_o^{-1}(e')$.

In addition, if Φ is also completely integrable, then, to any given path $\gamma_{e,e'}$ from e to e' in $\mathcal{M}^+$ there corresponds a unique, covering integral path $\hat{\gamma}$ in $E\mathcal{M}^+$ projecting on $\gamma_{e,e'}$ once its basepoint is given $\Gamma_o^{-1}(e)$.[8] Note that the 1-form α is not necessarily completely integrable and it appears to be the analogous of the *Weyl's length 1-form of connection*. In turn, $\mathcal{M}^+$ can be embedded univocally in $E\mathcal{M}^+$ as a pointed manifold (that the fibers are compact or not).

The compactness of the fibers are only required to be consistent with Ehresmann's theorem of 1947 (see proof in Appendix G) on the trivializations of fibrations [Ehr47a, Proposition 1, p.1611, satisfied on a manifold of class C^1][Ehr51, Proposition, p.31, on a manifold of class C^2 to insure the differentiation of 1-forms satisfying the Frobenius conditions], *i.e.*, to have a proper surmersion Γ_o. Instead of compact fibers, we could have $\mathcal{E}\mathcal{I}_o^+$ compact (since any continuous map from a compact space is proper) or Γ_o proper.

B. What is projective or conformal, and in which manifold?

Additionally, because $\mathcal{M}$ must have a conformal structure in the sense of Ehlers, Pirani and Schild [EPS72] or in the framework of the causal axiomatics [Zee64, KP67, Car71, Woo73, Mal77] (see Fig. II.1, p. 14), then, every geometrical object defined on $\mathcal{M}$ (or $\mathcal{M}^+$) is defined up to a conformal scaling.

---

[7] *i.e.*, the annihilator of Φ in $TE\mathcal{M}^+$ is projectable on $T\mathcal{M}^+$.

[8] This is presented also as the so-called "*Condition* (c)" in [Ehr51, see Definition in §3 p.36].

In particular, the conformal/scaling factors will be completely defined locally by five time stamps, *i.e.*, by a point τ in $E\mathcal{M}^+$. Then, $\mathcal{M}^+$ embedded in $E\mathcal{M}^+$ must be preserved by scalings in $E\mathcal{M}^+$, *i.e.*, $E\mathcal{M}^+$ must be equivalent, somehow, with respect to scalings in order to keep its conformal structure. In other words, $\mathcal{M}^+$ can be locally considered as a four-dimensional real projective space defined from $E\mathcal{M}^+$.[9] Then, $\mathcal{M}^+$ must be a *generalized Cartan space*, *i.e.*, it is *locally* homeomorphic to $\mathbb{R}P^4$ (and not only $\mathbb{R}^4$).[10] We note also that $\mathcal{M}^+$ cannot be globally homeomorphic to $\mathbb{R}P^4$ because its Euler-Poincaré characteristic is such that $\chi(\mathbb{R}P^4) = \chi(S^4)/2 = 1$, and from Geroch's theorem it must vanish to be a spacetime manifold [Ger67] (this result was also obtained formerly from different considerations by Ehresmann [Ehr43, see p.630] for $\mathbb{R}P^4$ and S^4, and in full generality in [Ehr47b, Corollaire, p.445]). Also, it is very important to note that if the manifold $E\mathcal{M}^+$ is at least of class C^2 and if it is also an open manifold then it can always admit a foliation of codimension 1.[11] And thus, in that case, we can always produce a local *projectivization* of $E\mathcal{M}^+$ to the generalized Cartan space $\mathcal{M}^+$.

Hence, we must consider that any geometrical object on $\mathcal{M}^+$ (metric, curvature, etc.) and in particular the functions ν_j must depend on five independent parameters, *i.e.*, the four time parameters τ_i, and some values depending on a path $\gamma_{o,e(\tau)}$ or an alternate fifth time stamp τ_5 coming from a fifth satellite. The latter will be introduced in a particular relativistic protocol of *localization* presented in the sequel and including the relativistic positioning protocol.

Additionally, because of the conformal structure on $\mathcal{M}$, we consider that another metric g at $e(\tau)$ differing from g at the same event $e(\tau)$ by a conformal factor only is just as valid to represent the underlying geometry of $\mathcal{M}$ as g. The example given above in the previous section appears to be a very fundamental example from which we can identify the different applications and define the general geometrical framework. In particular, we see from this

[9] Note that there exists a strong ambiguity in the terminology depending on whether we take the terminology used in physics or the one used in mathematics. Indeed, what the physicists call "conformal" is exactly what the mathematicians call "projective". Moreover, the projective structure of the spacetime in the sense of Ehlers, Pirani and Schild is associated, at least locally, with a three-dimensional projective space in the sense of the mathematicians rather than a four-dimensional one. Moreover, because a conformal structure involves to consider mathematically classes of metric fields defined up to conformal factors, it raises the question of the physical meaning of such conformal factors: are they void of physical meanings or, in fact, rather physically unreachable? In the designs of the relativistic positioning systems based on $\{\ell\ell\ell\ell\}$-frames, the charts of the atlas on $\mathcal{M}$ cannot be obtained without introducing such conformal factors revealed, for instance, by the shift parameters in the CFMT protocols. But, they appears to be themselves defined up to other conformal factors. Hence, rather than to be physically unreachable, an alternative could be that they are physically unreachable in an "absolute" sense. This latter aspect could be expressed in the necessary choice for a particular 'cut event', i.e., in the choice of a particular origin of the pointed space associated with $\mathcal{M}^+$ and embedded in $E\mathcal{M}^+$ and, moreover, the inability of the present relativistic positioning systems to obtain the "absolute" value of any conformal factor at any origin.

[10] $\mathbb{R}P^4$ is therefore locally "*tangent*" to $\mathcal{M}^+$ in the sense of Ehresmann.

[11] Indeed, codimension-one foliations on a C^2 manifold M can be defined from Morse functions on M which can be deformed, whenever M is open, to put their critical points at infinity [God91, p.9, Proposition 1.14]. On closed, connected differentiable manifolds M, the situation is more complex. In that case, to have a dimension-one or codimension-one foliation on M, then the Poincaré-Euler characteristic of M must vanish. This is a sufficient condition for any dimension-one foliation [Ste51, a result due to H. Hopf] and for any codimension-one foliation as well [Thu76].

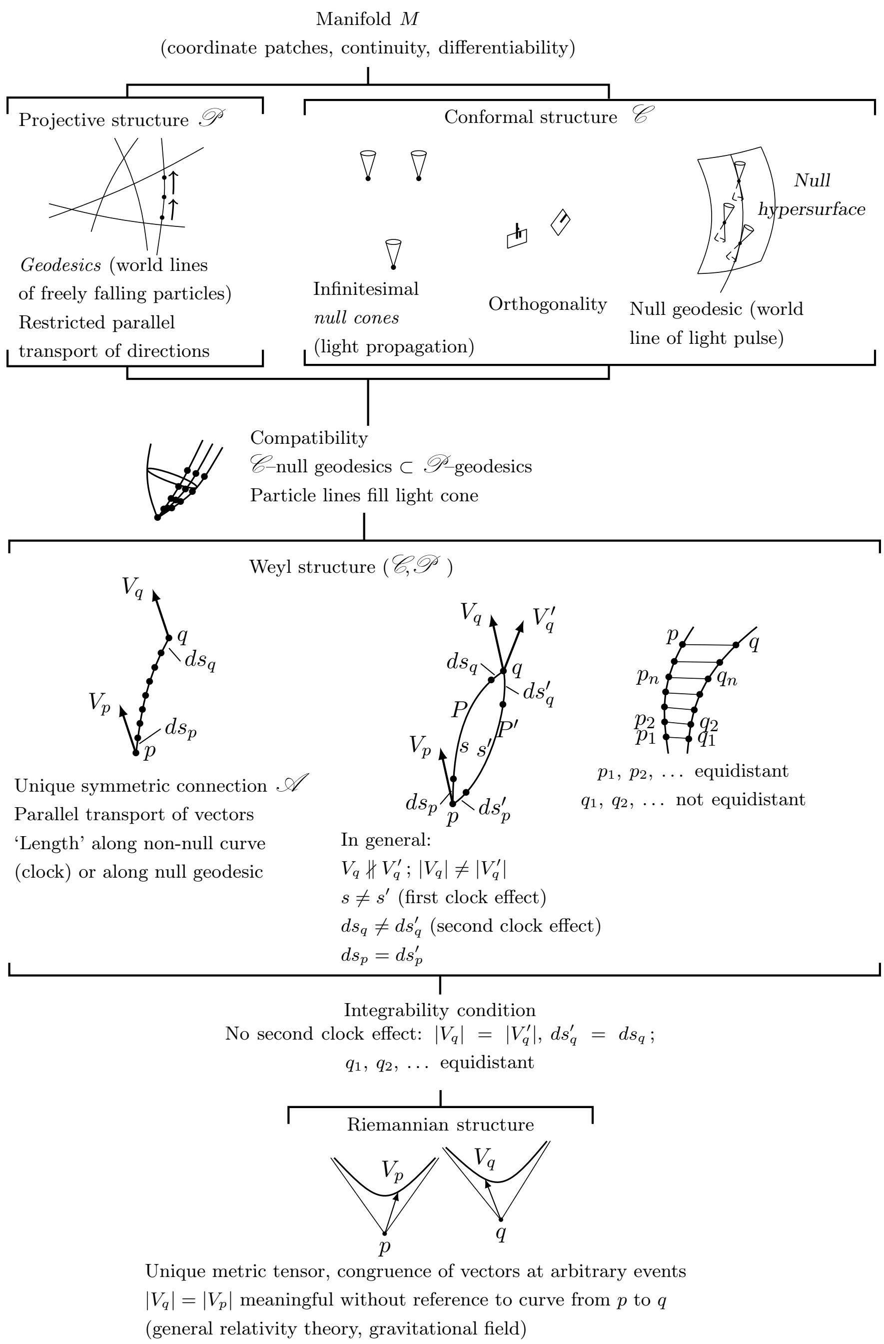


Figure II.1. General scheme of conformal, projective, Weyl, and Riemannian structures (From Fig. 1. in Ehlers-Pirani-Schild paper [EPS72, p.66]) example the '*homogeneous*' character of the metric field under scaling of the time parameters τ_i ($i = 1, \ldots, 4$) to new time parameters ς_i such that $d\varsigma_i = \lambda(\tau)\, d\tau_i$ where $\lambda \in \mathcal{O}_{E\mathcal{M}^+}$, much the same as there are *homogeneous* coordinates defining a point in a projective space.

Then, the metric g itself is scaled and transformed into $\mathrm{g} \equiv \lambda(\tau)^4\, g$. This scaling is consistent with changes of time parameterization due to a change of unit time standard used for the clocks embarked on the satellites (fundamentally, clocks are only *generators of events*, each characterized by an *identity number* increasing with time, and thus, not ascribed to follow or read univocally and rigidly as a *reading head* "*a*" time, somehow, integrated and stored on a spacetime tape or substrate. Besides, it indicates the ambiguous notion of time orientation of the spacetime which provides an absolute notion of time). This is a fundamental characteristic of the relativistic positioning systems and it follows that any geometrical object (metric, curvature, field equations, etc.) expressed with such charts must have this property of homogeneity with respect to any given set of four time stamps (τ_k or ς_k, etc.) enlarged with a fifth one.

Actually, É. Cartan shown that the projective tensors, such as g in the present context, differ deeply from the usual (affine[12]) tensors used in non-projective geometries [Car34, Car35, Car37].[13] For instance, the contravariant projective vectors (called historically *contravariant [analytic] vectors* by É. Cartan) are defined by a set of components which can be split in two parts, one of which, *i.e.*, the affine part, defines a contravariant (non-analytic) vector in the usual affine sense, *i.e.*, a contravariant vector with its origin at a point. This splitting cannot be obtained in full generality in the case of the covariant projective vectors because, contrary to affine tensors, contravariant and covariant projective vectors cannot be dual to each other.

Therefore, what the conformal equivariance on $\mathcal{M}$ involves, is to consider any (affine) tensor defined on $\mathcal{M}$ as the affine part of "larger" projective tensors. Hence, the goal is, in part, to extend most of the geometrical objects defined on $\mathcal{M}^+$ to larger objects defined mainly on $E\mathcal{M}^+$. As a result, the (five) time stamps must be seen as homogeneous coordinates of local vector spaces associated with local four-dimensional projective spaces, and then, any given event $e \in \mathcal{M}^+$ is, locally only, in a one-to-one correspondence with a point in a local projective space

[12] meaning tensors attached to points of a given manifold, rather than tensors all attached at the same common fixed origin point.

[13] Historically, these categories of projective tensors were entirely absent from the works of O. Veblen, B. Hoffmann, J. M. Thomas, J. A. Schouten, D. van Dantzig and J. Haantjes but noticed for the first time by É. Cartan.

$\mathbb{R}P^4 \cong_{loc.} [E\mathcal{M}^+]$ which is the set of vector lines $l \equiv [\tau] \simeq e \in \mathcal{M}^+$ (or $[\varsigma]$ where ς can be other homogeneous coordinates defined from the τ_α's) generated by the non-vanishing points $\tau \in E\mathcal{M}^+$.

Moreover, note that we cannot always deduce from this property of homogeneity on $\mathcal{M}$, another genuine, different and local projective geometry but three-dimensional without strong additional geometrical constraints, *i.e.*, at any given event $e \in \mathcal{M}$, we have a one-to-one local correspondence with a point in the product space $\mathbb{R}P^3 \times \mathbb{R}$.[14] Indeed, in this case, we could have at least the three-dimensional projective structure in the sense of Ehlers, Pirani and Schild, the definition of which is associated with the evaluations of the deviations of the timelike worldlines from the timelike geodesics. These deviations can be expressed by the so-called *geodesic spherical coordinates* which are the corresponding inhomogeneous coordinates of the local three-dimensional projective spaces associated with the space of velocity vectors rather than to the space of events. Then, reducing $\mathcal{M}$ locally to a three-dimensional projective space would mean that the whole of the geometrical objects defined on $\mathcal{M}$ would be, somehow, factorizable in two parts, one of which depending only on the geodesic spherical coordinates, and the other, depending only on one parameter, *e.g.*, a time parameter. Hence, a three dimensional projective space for the set $\mathcal{M}$ of events does not seem to be really conceivable contrary to the set of velocities.

Then, starting from $l \in \mathbb{R}P^4$ at an event $e \simeq l \equiv [\tau]$ $(e \in \mathcal{M}^+)$, the question remains also to design a particular physical and geometrical protocol *to recover* a complete τ-dependency of the different geometrical objects defined on $\mathcal{M}^+$ at this event, *i.e.*, we have to design a protocol breaking the local projective characteristic of the spacetime when introducing, possibly, certain path integrals or a fifth time stamp.[15] As a result, for instance, the connection on the manifold $\mathcal{M}^+$ will not be a Riemannian connection but a Cartan projective connection on $\mathbb{R}P^4$ and the use of a projective connection means that the geometry on $\mathcal{M}^+$ embedded (by the previous

[14] Note that the projective space $\mathbb{R}P^3$ is *homeomorphic* to $SO(3,\mathbb{R})$ and $SO(3,\mathbb{R})$ is *isomorphic* to $S^3/\{1,-1\}$ and S^3 is isomorphic to $\mathbb{H}_1$ which is the sub-group of *quaternions of unit norm* (see [God71, Proposition 3.8, p.40]).

[15] There is an other possibility: we can imagine having two charts, one based on a coordinate system $\tau \in \mathbb{R}^4$ and the other with the coordinate system $\tau' \in \mathbb{R}^4$, *i.e.*, we suppose there are two relativistic positioning systems and therefore two systems (constellations) of emitters. Then, we proceed as follows: we agree to identify these two charts (*i.e.*, "images") by matching each point τ of the first coordinate system with a point τ' of the other coordinate system if at these two positions in $\mathbb{R}^4$, corresponding to a unique event e in the spacetime $\mathcal{M}$, we have the same value of a given scalar density, *e.g.*, the determinant ("intensity") of g or a physical intensity of light for instance. Then, we can go back to, or "rebuild", the relative value of the conformal factor (path integral) at e common to both systems of coordinates. We deduce then, for example, the difference of the two functions $\phi(\mu)$ (in (2.11)) and $\phi(\mu')$. We can take anything else differing from the determinant of g such as for example the Riemannian scalar curvature or, better, the Weyl scalar curvature. To summarize, we could introduce a Morse theoretical aspect on $\mathcal{M}^2$, the latter containing then the embedded five-dimensional manifold $E\mathcal{M}^+$.

protocol) in $E\mathcal{M}^+$ will then be, somehow, "sensitive" physically to, for instance, the conformal factors and their variations. And then, we make effective the linkage of the physics with the geometry of the spacetime.

We can note also, to be exhaustive, that Haantjes, Hoffmann, Schouten, van Dantzig and Veblen proposed a projective theory of the relativity [VH30, Hof31, SvD32, SvD33, Sch33a, Sch33b, SH34, Sch35]. Their theory provides also a model of unification of the electromagnetism and the gravitation. The electromagnetic stress field, *i.e.*, the Faraday tensor, is obtained from the introduction of a nonvanishing torsion in the projective connection (see in [Sch35] the tensor of projective contorsion $S_{\mu\lambda}^{\cdot\cdot\chi}$, formula (53) p.67 and $\overset{R}{S}{}_{\mu\lambda}^{\cdot\cdot\chi}$, formula (72), p.71, and where "symmetric connection" means torsion-free. See also formula (76) for the projective connection tensor $\overset{R}{\Pi}{}_{\mu\lambda}^{\chi}$ and the projective connection $\overset{R}{\nabla}_{\mu}$ in formula (105). Also, the vector q can be identified with our notations to the vertical vector ξ defined in the sequel).

One of the main criticism made on their theory was that the homogeneous coordinates they introduced could not be linked clearly in any way to any physical parameters added, for instance, to those of space and time. But also, because the mathematical formalism they developed was too abstruse (K. Yano, supervised by É. Cartan, did his dissertation to clarify the formalism) and not unified among the mathematicians at this time.

In the present context, the homogeneous coordinates, *i.e.*, the five time stamps, are physical parameters which are clearly identified, and therefore, their unification theory appears to be really, strongly and physically admissible and validated by our relativistic localization protocol of which a detailed presentation will be given in the sequel. The projective connection exhausts all of the geometrical possibilities providing gravitation and electromagnetism fields unified in a unique projective (curvature) field of which the projective connection is the potential field.

Besides, we shown also that a causal representation of the metric g in a $\{\ell\ell\ell\ell\}$-coframe is first to any of its causal representations in a $\{ssst\}$-coframe (ℓ for light, s for space and t for time). Indeed, for any given $\{ssst\}$ representation of g in a given $\{ssst\}$-coframe there corresponds sometimes an infinite set—a loop homeomorphic to $S^1 \subset \mathbb{R}^4$—of $\{\ell\ell\ell\ell\}$ representations in $\{\ell\ell\ell\ell\}$-coframes (see details in Appendix C). We call this non-univocity a *loop degeneracy.* For instance, if g is represented by a diagonal matrix in a given $\{ssst\}$-coframe, with the diagonal coefficients β_j $(j = 1, \ldots, 4)$ fixed, then there is almost always an infinite set of corresponding $\{\ell\ell\ell\ell\}$-coframes, *i.e.*, an infinite set of coefficients ν_k $(k \neq 0)$.

Therefore, unfortunately, by matching the β's non-univocally with the ν's, we would define a fibration over $\mathcal{M}$ with, sometimes, loops as typical fibers from which the ν's would be associated with indefinite sections. Thus, we must absolutely use $\{\ell\ell\ell\ell\}$-coframes which are truly prior to any other kind of coframes, and therefore, we must forbid, for instance, the use of $\{ssst\}$-coframes in any intermediate computation.

Now, we recall some elements of the differential projective geometry and our notations which will be applied in the present context. Our approach is a combination of the Cartan and Ehresmann ones for reasons explained below.

III. A SHORT REVIEW ON THE PROJECTIVE DIFFERENTIAL GEOMETRY

A. Cartan versus Ehresmann approaches

Historically, to study the generalized projective geometry, there were three main different viewpoints among a lot of others: those of É. Cartan, T. Y. Thomas and D. van Dantzig. Latter, in his dissertation dedicated to "*his master, É. Cartan*" [Yan38], Yano linked all the different approaches to the one developed by É. Cartan in his 1924 seminal paper [Car24b].

One year later, in 1925, É. Cartan introduced his *generalized spaces* [Car25] which are nonholonomic versions of the homogeneous spaces (*i.e.*, homeomorphic to cosets of Lie groups). In this paper, the Cartan's goal was to build generalized spaces which are locally (infinitesimally) closed to homogeneous spaces, and then, to find a method to compare a generalized space to its homogeneous *model* (global) space.

In order to formalize and to understand better the global, *i.e.*, topological, viewpoint of these generalized spaces, Ehresmann published his fundamental paper on the infinitesimal connections in the differentiable fiber bundles [Ehr50, Ehr51]. A central excerpt from his paper is the following (a more modern approach is given in [Sha97, Appendix A, pp.357–373] or [AG93] for instance).

Let $E(B,F,G)$ be a differentiable fiber bundle of class C^2 with standard fiber F and structural Lie group G over the connected base manifold B. We denote by $E_x \subset E(B,F,G)$ any fiber over $x \in B$. The left action of G on F is assumed to be effective and transitive. Then, an infinitesimal connection C, called also an *Ehresmann connection* on $E(B,F,G)$ is defined from a differentiable transversal field C of contact elements of dimension $\dim B$ which satisfies 1) the so-called *Condition* (c) of Ehresmann [Ehr50, Lemme, p.154][Ehr51, Definition, §3 p.36]:

> Condition (c): *any differentiable path $\gamma_{x,x'} \subset B$ with basepoint x and endpoint x' is the projection of an integral curve $\hat{\gamma}_{z,z'}$ of C with basepoint $z \in E_x$ and endpoint $z' \in E_{x'}$; the point z being arbitrary in E_x,*

and 2) such that the homeomorphisms $\varphi_{\gamma_{x,x'}} : z \longrightarrow z'$ associated with the paths $\gamma_{x,x'}$ in the condition (c) are *isomorphisms* from E_x to $E_{x'}$.

Note that this condition (c) sounds strongly with the integration paths of the *Weyl's gauge of length* all the more so as Weyl worked also on the developments of the differential projective geometry [Wey56, Wey52]. But, it is also another expression of the notion of parallel transport along a curve. Also, this condition is obviously satisfied if C is completely integrable[16] in $E(B,F,G)$ which, therefore, becomes a foliated manifold. Additionally, if all the fibers E_x are compact (or, actually, $E(B,F,G)$ compact [Ehr50, Lemme p.154]), Ehresmann shown [Ehr47a, Proposition, p.1612] that any (completely integrable or not) transversal field C satisfies the condition (c) and any path $\gamma_{x,x'} \subset B$ defines a unique homeomorphism connecting E_x and $E_{x'}$.

Therefore, a notion of infinitesimal connection over $E/F \simeq B$ connecting a vector to a tangent bundle map can be somehow the infinitesimal version of a "global connection" φ connecting a path $\gamma_{x,x'}$ to a isomorphism $\varphi_{\gamma_{x,x'}}$.

Then, Ehresmann introduced the notions of *generalized Cartan spaces* (and Cartan connections, in particular, on projective spaces) that are connected manifolds B which must satisfy three conditions [Ehr51, §5 p.42]:

[16] Certain Frobenius conditions must be satisfied; so the need for $E(B,F,G)$ to be of class C^2.

- (c_1): *the typical fiber F must be a homogeneous space G/G' where G is the structural Lie group of $E(B,F,G)$ acting effectively and transitively on F and G' a closed Lie subgroup of G leaving invariant a given point $o \in F$.*

- (c_2): $\dim B = \dim F$.

- (c_3): *$E(B,F,G)$ has a section s of class C^2 embedding B into $E(B,F,G)$.*

Then, considering that G' is not a normal subgroup of G, $F = G/G'$ is the homogeneous *model space* (*e.g.*, $\mathbb{R}P^4$) of the *generalized space* B (*e.g.*, $\mathcal{M}$). Moreover, each fiber E_x is said to be *tangent* to B at $x \in B$. Also, the condition (c_3) can be cancelled out if B is identified with an embedded submanifold of $E(B,F,G)$, *i.e.*, B is an integrable manifold of a completely integrable transversal field C of contact elements. Then, $E(B,F,G)$ can have a structure of foliated manifold in a saturated tubular neighborhood of B.

For, in particular, from Reeb's theorem [Ree47, Theorem 2 with its complement], if $E(B,F,G)$ is of class C^2 and if B is compact and connected with a finite Poincaré group $\pi_1(B)$ then the neighboring leaves of B are compact and homeomorphic to coverings of B (and then, their respective Poincaré groups are subgroups of $\pi_1(B)$. Actually, these groups are all equal if the leaves are integral submanifolds). Similar results hold if B is embedded in $E(B,F,G)$ by a continuous section [Ehr44].

Furthermore, we can also associate with $E(B,F,G)$ a principal fiber bundle $P(B,G)$ with the same structural group G. For, we consider the manifold $F^N \equiv F\times\ldots\times F$ where N is the minimal integer such that the effective left action of G is also free on F^N. Also, we denote by h_z where $z \in B \subset E(B,F,G)$ any homeomorphism from F to the fiber $E_z \subset E(B,F,G)$. Consequently, we obtain the corresponding fiber bundle $E^N(B,F^N,G)$ with fibers $E^N_z \equiv \{h_z(F)\times\ldots\times h_z(F)\} \subset E_z\times\ldots\times E_z$ (N factors). Then, any homeomorphism h_z can be identified with an element of E^N_z (but not the converse).[17]

[17] Indeed, let h_z be such a given and fixed homeomorphism, then, the map $h_z \longrightarrow h^N_z \equiv h_z\times\ldots\times h_z$ (N factors) is bijective and continuous. Hence, if h^N_z is known, then h_z is known as well. Let L^g_z be the left action of $g \in G$ on E_z. We denote by $g.f \in F$ the left action of $g \in G$ on $f \in F$. Also, let $\tilde{h}_z$ be another homeomorphism such that $\tilde{h}_z = L^g_z \circ h_z = h_z \circ g$. Any homeomorphism $\tilde{h}_z$ from F to E_z can be written in this form because, first, we must have $L^g_z \circ h_z \circ g^{-1} \equiv Ad(g)h_z = h_z$ for all $g \in G$ and all homeomorphism h_z, and second, because the left action of G on the set of homeomorphisms h^N_z is free and transitive (because free and transitive on E^N_z and F^N). In this way, we can put in correspondence the image sets of each homeomorphism $\tilde{h}_z$, and then, each homeomorphism $\tilde{h}_z$. Thus, h_z being given and fixed, to each $g \in G$ corresponds an unique $\tilde{h}_z \in Hom(F, E_z)$ and reciprocally. And, moreover, to each $g \in G$ we can associate an unique element in E^N_z (or in F^N via h_z). Indeed, let $f^N \equiv (f_1,\ldots,f_N) \in F^N$ be fixed and given in addition to h_z such that

As a result, we can also identify $E^N(B, F^N, G)$ with the principal fiber bundle $P(B, G)$ over B defined as the set of all the homeomorphisms h_z between F and the fibers E_z where $z \in B \subset E(B, F, G)$. Roughly speaking, given two elements h_z and h'_z in $P(B, G)$, then $h_z \circ (h'_z)^{-1} \in Aut(E_z)$ is a local representation of an element of the standard fiber of $P(B, G)$ which is the structural group G. The right action of G on the homeomorphisms $h_z : F \longrightarrow E_z$ corresponding to the left action of G on F is then free and transitive on the fibers H_z of $P(B, G)$ which are the sets of homeomorphisms h_z. In addition, it follows that the Lie algebra $\mathcal{G}$ of G is isomorphic to each vertical vector space $T_{h_z} H_z$ tangent at $h_z \in H_z$ to the fibers H_z of $P(B, G)$.

We can notice that the frame bundles $P(B, G)$ associated with the bundles $E(B, F, G)$ admit always left G-invariant vector fields, and thus, there exists always an *integral* infinitesimal connections $\overline{C}$ on the principal bundles $P(B, G)$, and then, which satisfy the condition (c). From these infinitesimal connections $\overline{C}$, we can associate other *integral* infinitesimal connections C on $E(B, F, G)$ satisfying also the condition (c) via the map: $h_z \in P(B, G) \longrightarrow h_z(f) \in E(B, F, G)$ where $f \in F$ is fixed [Ehr50, Proposition, p.160][Ehr51, Proposition, p.39].

Moreover, this principal bundle can be reduced to a principal sub-bundle $P'(B, G')$ with structural group G' and associated with a fiber sub-bundle $E'(B, F, G')$ which is said to be *soldered* to $B \subset E'(B, F, G')$. The bundle $P'(B, G')$ is the set of all homeomorphism h'_z such that $h'_z(o) = z$ for all $z \in B$. We consider that G' is identified with the isotropy group of the origin o, and then, it is easy to see that the right action of G' preserves[18] such sets of homeomorphisms h'_z. Additionally, because the left action of G is effective (and transitive) on F, then, the right action of G' on the homeomorphisms h'_z is free (and transitive), and thus, H'_z and G' are diffeomorphic manifolds. This is also true for G, *i.e.*, the right action of G on the homeomorphisms h_z is free (and transitive), and thus, but also by definition, the fibers H_z of $P(B, G)$ and G are diffeomorphic manifolds. (see the previous footnote 17, p. 20).

all the f_i $(i = 1, \dots, N)$ are distinct. Thus, if we denote by $Iso(f) \subset G$ the isotropy group associated with $f \in F$, then, we have $\cap_{i=1}^N Iso(f_i) = \{id\}$. Now, let $A : G \longrightarrow E_z^N$ be the continuous map such that $A(g) = \tilde{e}_z^N \equiv (\tilde{e}_1, \dots, \tilde{e}_N)$ where $\tilde{e}_i = h_z(g.f_i)$ for $i = 1, \dots, N$. Then, A is injective. Indeed, if $\tilde{e}_i = h_z(g.f_i) = h_z(g'.f_i)$ for all $i = 1, \dots, N$ where $g \neq g'$ (and thus, $g'' = g^{-1}.g' \neq id$), then, for all $i = 1, \dots, N$ we have $h_z(g.f_i) = h_z(g.g''.f_i) = h_z(g.f'_i)$ where $f'_i = g''.f_i$. Thus, if we set $\tilde{h}_z(f) = h_z(g.f)$ for all $f \in F$, then $\tilde{h}_z(f_i) = \tilde{h}_z(f'_i)$ for all $i = 1, \dots, N$. But, $\tilde{h}_z$ is a homeomorphism, and consequently, we deduce that $f_i = f'_i = g''.f_i$ for all $i = 1, \dots, N$. Hence, $g'' \in \cap_{i=1}^N Iso(f_i)$, and thus, $g'' = id$, and $g'' \neq id$; a contradiction. We can note that A is not surjective in full generality. As a result, for all $\tilde{h}_z$ such that $\tilde{h}_z(f) = h_z(g.f)$ there corresponds an unique element in E_z^N (or in F^N via h_z), but the converse is false because A is not surjective.

[18] Let $g' \in G'$ and the right action $R_{g'}$ of g' on $h'_z \in H'_z$. We have $R_{g'} h'_z = \tilde{h}_z$, *i.e.*, for all $f \in F$, we have $(R_{g'} h'_z)(f) = h'_z(g' f) = \tilde{h}_z(f)$. Then, first, we obtain a different homeomorphism $\tilde{h}_z(f) \neq h'_z(f)$, and second, $\tilde{h}_z(o) = h'_z(g' o) = h'_z(o) = z$, and therefore, we have also $\tilde{h}_z \in H'_z$.

Moreover, we have (see footnote 17, p. 20) homeomorphic correspondences $h_z \in H_z \longleftrightarrow (f_1, \ldots, f_N) \equiv f^N \in F^N/\triangle^N$ ($\triangle^N$ is the diagonal set[19]) from which we deduce, in particular for $h_z \equiv h'_z \in H'_z$, that any infinitesimal variation $\delta h'_z \in T_{h'_z} H_z \, (\supset T_{h'_z} H'_z)$ of h'_z is projectable to a tangent vector $(0, \ldots, 0, \mathfrak{g}.f_N) \in T_{f^N} F^N$ where $\mathfrak{g} \in \mathcal{G}$ (where $\mathcal{G}$ is the Lie algebra of G). Indeed, G acts transitively on F, and therefore, we can always find $N-1$ points $f_i \in F$ such that $\mathfrak{g}$ is an element of all of the Lie algebra of the isotropy groups $Iso(f_i)$ (and thus $\mathfrak{g}.f_i = 0$) but necessarily with $\mathfrak{g}.f_N \neq 0$. In particular, considering that the tangent spaces $T_f F$ are isomorphic, we can associate with $\mathfrak{g}.f_N \in T_{f_N} F$ an isomorphic tangent vector in $T_o F$. It follows that *any nonvanishing* vector $\delta h'_z$ in any tangent space $T_{h'_z} H_z$ is a vector projectable to a always *nonvanishing* tangent vector in $T_o F$.

Also, $P'(B, G')$ can be identified to a principal sub-bundle $E'^{N-1}(B, F^{N-1}, G') \subset E^N(B, F^N, G)$ since we can define univocally an element of $P'(B, G')$ taking f_N in the sequence f^N such that $f_N \equiv o$ which is the origin of F preserved by G'.

From, Ehresmann defined then the notion of *Cartan connection of type F over B* (called also a *Ehresmann connection*) as follows [Ehr51, see Definition, §5 p.43]:

> Definition. *Let $E(B, F, G)$ be a fiber bundle satisfying the conditions (c_1) and (c_2) and where the base manifold $B \subset E(B, F, G)$ is connected. Then, a Cartan connection $\mathfrak{w}$ of type F over B is a $\mathcal{G}$-valued 1-form defined on $TP'(B, G')$ such that*
>
> 1. *$\forall \mathfrak{g}' \in \mathcal{G}'$ and $\forall h'_z \in P'(B, G')$ then we must have*
>
> $$i_{\mathfrak{g}'^*} \mathfrak{w}(h'_z) \equiv \langle\, \mathfrak{w}(h'_z) \,|\, \mathfrak{g}'^*_z \,\rangle = \mathfrak{g}' \,, \tag{3.1}$$
>
> *where i is the interior product, $\mathfrak{g}'^*_z \in T_{h'_z} H'_z \subset T_{h'_z} P'(B, G')$ is the canonical, right invariant, vertical vector at h'_z associated with $\mathfrak{g}'$, i.e., $\mathfrak{g}'^*_z : F \longrightarrow T_{h'_z(\,.\,)} E_z$ is such that*
>
> $$\mathfrak{g}'^*_z \equiv R_{\exp(-t\,\mathfrak{g}')} \frac{d}{dt} R_{\exp(t\,\mathfrak{g}')}(h'_z)\Big|_{t=0} \,, \tag{3.2}$$

[19] $\triangle^N$ is the closed set of elements $(f_1, \ldots, f_N) \in F^N$ such that, at least, two elements f_i are equals (see footnote 17, p. 20).

with $\mathfrak{g}'^*_z(o) = 0 \in T_{h'_z(o)}E_z \equiv T_z E_z$ *and where* R_g *is the right action of* $g \in G$ *on* h'_z,

2. $\forall g \in G$ *and* $\forall \mathfrak{v} \in T_{h'_z}H'_z$ *then we must have*

$$i_{\mathfrak{v}} R^*_g(\mathfrak{w})(h'_z) \equiv \langle \mathfrak{w}(R_g(h'_z)) \,|\, T_{h'_z} R_g(\mathfrak{v}) \,\rangle = Ad(g^{-1})\, i_{\mathfrak{v}}\mathfrak{w}(h'_z)\,, \tag{3.3}$$

i.e., $\mathfrak{w}$ *is called a* ‘pseudo-tensorial 1-form of type Ad,’ *and,*

3. *if* $i_{\mathfrak{v}}\mathfrak{w} = 0$ *for* $\mathfrak{v} \in T_{h'_z}H'_z$ *then* $\mathfrak{v} = 0$. *Equivalently,* $\mathfrak{w}$ *is injective on* TH'.

Then, because B is embedded in $E'(B,F,G') \subset E(B,F,G)$, and thus, such that each point $z \in B$ is in a one-to-one punctual correspondence with $o \in F$ via the homeomorphisms h'_z, we must, somehow, *solder* locally either F or each H'_z to B more strongly. In other words, any local chart containing $o \in F$ (or local chart of H'_z containing $z \in B$) must also be used as if it is a local chart of B containing the point z. Thus, in particular, each tangent space T_oF must be put biunivocally in correspondence with the tangent space T_zB.

For, we notice that any *nonvanishing* infinitesimal variation $\delta h'_z \in T_{h'_z}H_z$ is projected on a *nonvanishing* vector in T_oF. Additionally, because G acts transitively and freely on each fiber H_z of $P(B,G)$, then all the infinitesimal variations $\delta h'_z$ generate $T_{h'_z}H_z$. Furthermore, because G' act freely and transitively on the h'_z and because $H'_z \simeq G'$, then all the infinitesimal right actions of G' on the homeomorphisms h'_z generate also the tangent space $T_{h'_z}H'_z$.

Then, let R_g be the right action of $g \in G$ on $h'_z \in H'_z$, *i.e.*, such that $R_g(h'_z)(f) = h'_z(g.f)$ for all $f \in F$. Moreover, if $g \equiv g' \in G'$ then $R_{g'}$ defines a diffeomorphism $L_{g',z}$ of the fiber E_z such that $L_{g',z}(e_z) = e'_z$ where $e_z \equiv h'_z(f)$ and $e'_z \equiv h'_z(g'.f)$. Now, there always exist G'-valued $Ad(G)$-invariant frames in the *principal* bundle $P(B,G)$. Each is associated with an *integrable* Pfaff system $\widehat{C}$ of n $\mathcal{G}'$-valued $ad(\mathcal{G})$-invariant 1-forms on $P(B,G)$. Given a smooth bundle (map) embedding from $E(B,F,G)$ to $P(B,G)$ provides by pull-back an *integrable* Pfaff system C of n $\mathcal{G}'$-valued $Ad(\mathcal{G})$-invariant Pfaff system of 1-forms on $E(B,F,G)$ which represent the infinitesimal actions of $L_{g',z}$ and a field of projectors $\mathfrak{Q}_z$ on the tangent spaces TE_z of the fibers E_z. Therefore, we can also deduce a field of supplementary projectors $\mathfrak{P}_z$ from $T_zE(B,F,G)$ onto T_zB.

Besides, let g be an element of G and $h'_z \in H'_z$. Then, we define the homeomorphism $\tilde{h}$ such that $\tilde{h} = R_g(h'_z)$. Clearly, if, moreover, $g \in G'$, then we have $\tilde{h}(o) = h'_z(g'.o) = h'_z(o) = z$ and thus $\tilde{h} \equiv \tilde{h}_z \in H'_z$. On the contrary, if $g \in G/G'$ then $\tilde{h} \in H_{\tilde{z}}$ with $\tilde{z} \notin E_z$. Thenceforth, we consider the infinitesimal variation $\delta\tilde{h}$ of $\tilde{h}$ due to an infinitesimal variation $\delta g = \mathfrak{g}\, g$ of g where $\mathfrak{g} \in \mathcal{G}/\mathcal{G}'$. We obtain $\delta\tilde{h}(f) = T_f h'_z(\mathfrak{g}.f) = \delta z$ and, in particular, $\delta\tilde{h}(o) = T_o h'_z(\mathfrak{g}.o) = \delta\tilde{z} \notin T_{\tilde{z}}E_{\tilde{z}}$. Also, in this case we can notice that $\delta\tilde{h} \simeq \delta h'_z$, $\tilde{z} \simeq z$ and $\delta\tilde{z} \simeq \delta z$ if $\mathfrak{g}$ is sufficiently closed to an element $\mathfrak{g}' \in \mathcal{G}'$ relative to the Euclidean canonical norm (and with an abuse of notation we can write also $\delta\tilde{h} \equiv \delta h'_z \in T_{h'_z}H_z$ in full generality). Therefore, we obtain $\mathfrak{P}_{\tilde{z}}(\delta\tilde{h}(o)) \equiv \delta_B\tilde{z} \neq 0$ and $\mathfrak{Q}_{\tilde{z}}(\delta\tilde{h}(o)) \equiv \delta_E\tilde{z} \neq 0$ with, respectively, $\delta_B\tilde{z} \in T_{\tilde{z}}B$ and $\delta_E\tilde{z} \in T_{\tilde{z}}E_{\tilde{z}}$.

Furthermore, we can define two surjective maps w_g and u_g where $g \in G/G'$ and where $\tilde{f} \in F$ is such that $\tilde{f} = \tilde{h}^{-1}(\tilde{z})$ and $w_g : (\tilde{z}, \mathfrak{g}) \in B \times \mathcal{G}/\mathcal{G}' \longrightarrow (\tilde{z}, \mathfrak{P}_{\tilde{z}}(\delta\tilde{h}(o))) \in TB$ and $u_g : (\tilde{z}, \mathfrak{g}) \in B \times \mathcal{G}/\mathcal{G}' \longrightarrow (\tilde{h}^{-1}(\tilde{z}), T_{\tilde{z}}\tilde{h}^{-1} \circ \mathfrak{Q}_{\tilde{z}}(\delta\tilde{h}(o))) \in TF$. They are also homeomorphisms since only nonvanishing $\delta\tilde{h}(o)$ project onto nonvanishing vectors in $T_{\tilde{f}}F \simeq T_oF$ and $T_{\tilde{z}}B$ and also because the dimensions of the source and image vector spaces are equal.

Then, there exists [Ehr51, see first Proposition, §5 p.43][Ehr50, see first Proposition, p.162] a *soldering* homeomorphism S_g factorizing w_g, *i.e.*, such that the diagram

$$\begin{tikzcd} B \times \mathcal{G}/\mathcal{G}' \arrow[r, "w_g"] \arrow[d, "u_g"'] & TB \arrow[dl, dashed, "S_g"] \\ TF & \end{tikzcd} \tag{3.4}$$

commutes because TF is an injective module.

Then, a conclusion is that $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}/\mathcal{G}'$ act freely on, respectively, TH, TH' and TF (via u_g), but there does not exist a free action of $\mathcal{G}/\mathcal{G}'$ on TH'. This forbids to built a $\mathcal{G}$-connection on TH. Only a $\mathcal{G}'$-connection on TH' exists which is the Ehresmann connection $\mathfrak{w}$. Moreover, we saw that $TH' \simeq (TF)^{N-1}$ and we have also $TH' \simeq (TB)^{N-1}$ from the soldering map S_g. Hence, from the Ehresmann connection $\mathfrak{w}$ we can deduce isomorphically from S_g a $\mathcal{G}'$-connection ω on $(TB)^{N-1} \simeq P'(B, G')$ given a $g \in G/G'$. As a result, we deduce also that the set $\{\omega\}$ of Cartan connections is homeomorphic to $G/G' \simeq B$.

However, although Ehresmann provided a beautiful, great summarized construction with essential, fundamental results allowing to gather topology, foliations theory and fiber bundle descriptions, to approach powerfully in an unique framework a lot of geometries as diverse as the affine, Euclidean, projective, conformal, almost complex or element of contact structures, Cartan's approaches remain very useful in practice to implement such geometrical structures. In particular, referring to the projective geometry for instance, starting from Lie group representations is not needed in Cartan's approach contrary to what Ehresmann's construction suggests strongly. Yet, É. Cartan knew is own Maurer-Cartan theory…, but he did not use it to construct his projective connections. Actually, he utilized only these Lie group aspects explicitely to feature the so-called *projective 'analytic' tensors* he introduced in his other seminal 1935 Moscow paper [Car35].

For our part, we need only to start with the basic ground idea that we must describe some connections between punctuated sets of congruences of vector lines from which Lie group aspects result only, in some way, as (nevertheless very important) "residues." This is very useful if we want to link the physics and the geometry since lines can be related to paths of light rays for instance (and the Thomas's projective geometry of "paths" might be a mark of the possible optical physical origins of this geometry).

In his 1924 seminal paper on projective connections, É. Cartan used clever, powerful and useful tricks which cannot be easily set aside or ignored in practice. He never considered Lie group representations in his developments on the determination of a projective connection. Moreover, É. Cartan introduced the notions of *torsion-free* and *normal* projective connections which are not completely discussed by Ehresmann except, a little, the integrable Cartan connections and their corresponding vanishing projective curvatures [Ehr51, see §7 p.49–50].

In fact, Cartan provided a method to obtain "normal forms" for projective connections that we call *projective Cartan connections.* And the latter differ from those spelled with the same terminology by Ehresmann in, for instance, his definition (given p. 23) of (projective) *Cartan connections* in *generalized Cartan spaces*, and which are, actually, the most general (projective) connections. Then, the status of such projective Cartan connections (given by Cartan) with respect to the projective connections in general is analogous to the one between square matrices and Jordan forms. More exactly, the projective Cartan connections are to the projective connections what the Jordan forms are to the square matrices. Ehresmann never discussed about such "normal" forms for the (projective) connections.

Another disadvantage of the Ehresmann's approach is that no practical indications are given to reach the Cartan connection $\mathfrak{w}$ from its restricted Maurer-Cartan 1-form for G', *i.e.*, no indications are given on how to extend this restricted 1-form to $\mathfrak{w}$. On the contrary, the great advantage of the Cartan's approach is that É. Cartan started explicitly from the whole of the connection. This extension aspect is often put aside in a lot of works on projective connections.

Unfortunately, Cartan's approach on projective connections is based on a trivial foliation, *i.e.*, a foliation of $\mathbb{R}^{n+1}$ by affine hyperplanes $\mathbb{R}^n$. Yano pointed out this difficulty in his 1938 dissertation when he compared the different historical formalisms at this time for the differential projective geometry [Yan38, dissertation in french]. Nevertheless, he did not really introduce the notion of integrable codimension-one foliations which would have extended the Cartan's formalism.

Hence, in what follows, we build a Cartan projective connection modifying slightly the original work of É. Cartan to take into account a non-trivially foliated manifold. In the same time, this is an opportunity to present the original Cartan's paper of which no english translations exist to the author's knowledge. Also, the modified Cartan's method we give in the sequel could be an intermediate manner to present the differential projective geometry to non-experts not aware of the modern geometrical formalism, all the more so as we present a fundamental and physical application of this geometry.

B. The projective manifold and the projective group actions

Any projective manifold $\mathbb{R}P^n$ is homeomorphic to the affine set of points $P_0 \cup \mathbb{R}P^{n-1}$ where the n-dimensional affine space $P_0 \subset \mathbb{R}^{n+1}$ is homomorphic (and thus, not only homeomorphic) to the vector sub-space $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ as soon as an 'origin' for P_0 is given, *i.e.*, a particular nonvanishing vector in $\mathbb{R}^{n+1}$. Moreover, in this union, $\mathbb{R}P^{n-1}$ is homeomorphic to the $n-1$-dimensional projective space defined on the n-dimensional affine space defined by P_1 parallel to

P_0 (*i.e.*, $P_0 \cap P_1 = \varnothing$ in the Euclidean space $\mathbb{R}^{n+1}$). The projective (affine) space $\mathbb{R}P^n$ inherits a quotient topology but it can be endowed also with the topology of a so-called '*topological projective plane*'[20] in which open sets of its topology are such that the *join* and *intersection maps* are continuous.[21]

Actually, we have what we call a fundamental '*projective decomposition*' and two supplementary projectors $\mathbf{p}_0$ and $\mathbf{q}_0$ defined on the vector space $\mathbb{R}^{n+1}$ to describe P_0 and/or P_1. The formers are obtained and defined from a given dual form[22] π_0 on $\mathbb{R}^{n+1}$ we call the '*projecting form*' or the Yano-Ishihara 1-form [YI67, denoted by $\tilde{\eta}$] and a given vector $\xi_0 \in \mathbb{R}^{n+1}$ such that $\pi_0(\xi_0) = 1$; and then, we obtain

$$\mathbf{p}_0 \equiv \xi_0 \otimes \pi_0\,, \qquad\qquad \mathbf{p}_0 + \mathbf{q}_0 = \mathbb{1}\,. \tag{3.5}$$

A '*point*' $[\eta] \in \mathbb{R}P^n$ is a vector line $[\eta]$ of the vector space $\mathbb{R}^{n+1}$ generated by the nonvanishing vector $\eta \in \mathbb{R}^{n+1}$, and thus, $[\eta] \subset \mathbb{R}^{n+1}$. Then, there exists a vector $\overrightarrow{op} \in [\eta]$ (where o is the origin of the Euclidean space $\mathbb{R}^{n+1}$) with the unique decomposition:

$$\overrightarrow{op} = k\,\xi_0 + \overrightarrow{o\underline{p}} \in \mathbb{R}^{n+1}\,, \ \text{(or, equivalently, such that } p = k\,\xi_0 + \underline{p} \in \mathbb{R}^{n+1}) \tag{3.6}$$

where $k = 0,\, 1$ and $\pi_0(\overrightarrow{o\underline{p}}) = 0$, *i.e.*, $\overrightarrow{o\underline{p}} \in P_1$ considering P_1 as a vector space. Hence, a vector $\overrightarrow{op}$ with such decomposition and $k = 1$ defines a '*point*' p of the affine space $P_0 \subset \mathbb{R}^{n+1}$.

Furthermore, an origin must be given to define completely $\mathbb{R}P^n$, that is a point $s_0 \in P_0$, and thus, such that $\overrightarrow{os_0} - \xi_0 \in P_1$. Then, given P_1 and the origin such that $\overrightarrow{os_0} \notin P_1$, P_0 is defined from the condition $P_0 \cap P_1 = \varnothing$ in the Euclidean space $\mathbb{R}^{n+1}$.

Additionally, a 'projective frame' $\mathit{\Psi} \equiv \big\{[\zeta_0], \ldots, [\zeta_{n+1}]\big\}$ on $\mathbb{R}P^n$ is a set of $n+2$ distinct vector lines, or points $[\zeta_\alpha] \in \mathbb{R}P^n$ ($\alpha = 0, \ldots, n+1$) such that 1) the vectors ζ_β for $\beta = 0, \ldots, n$ form a basis of $\mathbb{R}^{n+1}$, and 2) ζ_{n+1} is defined by the relation:

$$\zeta_{n+1} = \sum_{\alpha=0}^{n} \zeta_\alpha\,. \tag{3.7}$$

[20] In the sense of Salzmann [Sal57, Buc79, Zan94, Sch02, Mck05].

[21] From any two distinct points p_1, $p_2 \in P_0$, the *join map* associate the unique '*point*' $p_1p_2 \in P_1$ (*i.e.*, affine line $p_1p_2 \subset P_0$), and from any two distinct '*points*' λ_1, $\lambda_2 \in P_1$ (*i.e.*, affine lines λ_1, $\lambda_2 \subset P_0$) the *intersection map* associate the unique common point $\lambda_1 \cap \lambda_2 \in P_0$, and there are at least 4 points, no three of which are on the same line.

[22] In other words, π_0 is a linear map from the vector space $\mathbb{R}^{n+1}$ to $\mathbb{R}$, or a constant differential 1-form on $\mathbb{R}^{n+1}$.

Moreover, a particular n-tuple is attached in a conventional way (well-justified in the sequel) to each vector $\zeta_\beta \in \mathbb{R}^{n+1}$ of Ψ or, equivalently, to each vector line $[\zeta_\beta] \subset \mathbb{R}^{n+1}$. The n-tuples attached to the vector lines $[\zeta_i]$, or the vectors ζ_i as well, for $i = 1, \ldots, n$, are the ordered sequences $[0, \ldots, 0, +\infty_i, 0, \ldots, 0]_n$ of lenght n where the ∞ symbol is at the i-th position from the left. They "represent" the points $[\zeta_i] \in \mathbb{R}P^n$. In a way, these n-tuples are "*projective weights*",[23] the same attached to each vector in the same vector line $[\zeta_i]$ in $\mathbb{R}^{n+1}$, and thus, they can be ascribed to "affine projective coordinates."

Furthermore, by convention, we consider that the points $[\zeta_\alpha]$ of Ψ are represented by the following pairs: $[\zeta_i] \equiv (p_i, [0, \ldots, 0, +\infty_i, 0, \ldots, 0]_n)$ for $i = 1, \ldots, n$, $[\zeta_0] \equiv (p_0, [0, \ldots, 0]_n)$ and $[\zeta_{n+1}] \equiv (p_{n+1}, [1, \ldots, 1]_n)$ where $\zeta_i \equiv \overrightarrow{op_i} \in P_1$, $\{p_0\} = P_0 \cap [\zeta_0]$ and $\{p_{n+1}\} = P_0 \cap [\zeta_{n+1}]$.

The projective frame Ψ yields a coordinate chart for $\mathbb{R}P^n$. Indeed, from a vector $\eta \in \mathbb{R}^{n+1}$ such that $\eta \notin P_1$ and corresponding to a point $[\eta] \in \mathbb{R}P^n$, we can obtain the n-tuple representing $[\eta]$. It can be determined from the projective frame Ψ in the following way (see Figure III.1).

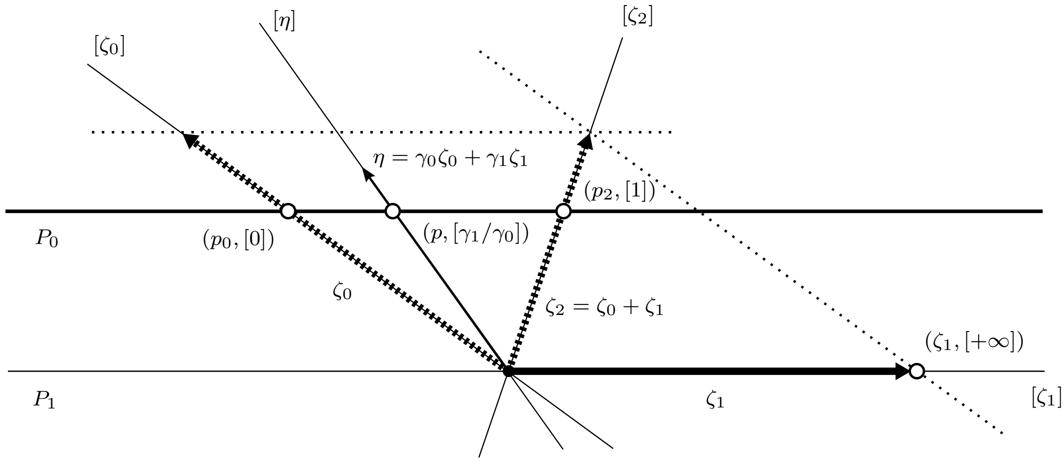


Figure III.1. Schema representing the projective frame $\Psi = \{(p_0, [0]), (\zeta_1, [+\infty]), (p_2, [1])\} \equiv \{[\zeta_0], [\zeta_1], [\zeta_2]\}$ of the projective space $\mathbb{R}P^1$ in $\mathbb{R}^2$.

First, to each pair $(p_i,\ [0, \ldots, 0, +\infty_i, 0, \ldots, 0]_n) \in \Psi$ we associate a vector $\beta_i\, \overrightarrow{op_i} \in P_1$ where $\beta_i \in \mathbb{R}^*$. Second, similarly, we associate respectively with $(p_0, [0, \ldots, 0]_n)$ and $(p_{n+1}, [1, \ldots, 1]_n)$

[23] A terminology used by É. Cartan, 1924, footnote n° 2, p.209 [Car24b].

a vector $\beta_0\,\overrightarrow{op}_0 \in \mathbb{R}^{n+1}$ and a vector $\beta_{n+1}\,\overrightarrow{op}_{n+1} \in \mathbb{R}^{n+1}$ with $\beta_0\beta_{n+1} \neq 0$. Then, we modify the coefficients β_i in such a way to satisfy the equality (3.7), *i.e.*, $\beta_{n+1}\,\overrightarrow{op}_{n+1} = \sum_{k=0}^{n} \beta_k\,\overrightarrow{op}_k$. Once the latter is satisfied for a particular set of coefficients β_α, up to a common factor, we define the ζ_α's such that $\zeta_\alpha \equiv \beta_\alpha\,\overrightarrow{op}_\alpha$ for $\alpha = 0, \ldots, n+1$. Then, we have the following unique decomposition for $\eta \notin P_1$: $\eta = \sum_{\alpha=0}^{n} \gamma^\alpha\,\zeta_\alpha$ with $\gamma^0 \neq 0$, from which we deduce its associate n-tuple $[\kappa^1, \ldots, \kappa^n]_n$ where $\kappa^i \equiv \gamma^i/\gamma^0$, *i.e.*, we obtain from the homogeneous coordinates γ^α the projective inhomogeneous coordinates κ^i in the projective frame $\mathit{\Psi}$ of a point $p \in P_0$ corresponding to $[\eta]$.

In a way, the point p_{n+1} is a sort of ruler for the definition of a projective frame since once the $n+1$ points p_α are given and fixed then any variation of p_{n+1} makes a redefinition of the coefficients β, and as a result, of the κ's; and thus, a redefinition of the projective frame with different values for the κ's ascribed to the same point $[\eta]$.

This construction is consistent and justify the initial convention given above for defining the n-tuples associated with each element of the projective frame $\mathit{\Psi}$. Actually, $[\eta]$ is such that $[\eta] \equiv (p, [\kappa^1, \ldots, \kappa^n]_n)$ where $p = \xi_0 + \underline{p} \in P_0 \cap [\eta]$ if $\eta \notin P_1$, and $\overrightarrow{op} \equiv \eta \in P_1$ otherwise. We can say that the affine manifold P_0 is a local '*representing manifold*' of $\mathbb{R}P^n$, and p represents $[\eta]$ in a local chart with the system of inhomogeneous coordinates $[\kappa^1, \ldots, \kappa^n]_n$. If $\eta \in P_1$, *i.e.*, $[\eta] \in \mathbb{R}P^n$ is a point 'at infinity', we proceed recursively from the dimension of the projective space, *i.e.*, we start with $[\eta] \in \mathbb{R}P^{n-1}$ and we consider a projective frame made of the vector lines $[\rho_i] \equiv [\zeta_{i+1}] \subset P_1$ with $i = 0$ to $n-1$ only, $\rho_n = \sum_{k=0}^{n-1} \rho_k$ and $n-1$-tuples.

Thus, we obtain a bijective correspondence between, in particular, $p \in P_0$ and $[\eta] \in \mathbb{R}P^n$ wether $p \in P_0 \cap [\eta] \neq \varnothing$ and then, accordingly, ***we will use more generally one or the other notation*** p ***or*** $[\eta]$ ***to designate any point of*** $\mathbb{R}P^n$***.***

Also, given a projective frame $\mathit{\Psi} \equiv \{\ell_0, \ldots, \ell_{n+1}\}$ where the ℓ's are vector lines in $\mathbb{R}^{n+1}$, then we can deduce, up to a scaling factor, a unique Euclidean frame $\mathit{\Phi} \equiv \{\zeta_0, \ldots, \zeta_n\}$ such that $\zeta_\alpha \in \ell_\alpha$ ($\alpha = 0, \ldots, n$) and $\sum_{\alpha=0}^{n} \zeta_\alpha \in \ell_{n+1}$. In other words, $\mathit{\Psi}$ is univocal to a "conformal" class of Euclidean frames $\mathit{\Phi}$: $\mathit{\Psi} \equiv [\mathit{\Phi}]$. Conversely, given a Euclidean frame $\mathit{\Phi}$ we can deduce the corresponding projective frame by the formula $\mathit{\Psi} \equiv \{[\zeta_0], \ldots, [\zeta_n], [\zeta_0 + \ldots + \zeta_n]\}$.

Additionally, we notice that a change of Euclidean frame in $\mathbb{R}^{n+1}$ can be associated with a change of projecting form π_0 (or ξ_0), *i.e.*, a change of projectors $\mathbf{p}_0$ and $\mathbf{q}_0$ and of projective decomposition.

Besides, the projective linear group $PGL(n+1,\mathbb{R}) = GL(n+1,\mathbb{R})/\mathbb{R}^*$ acts *transitively* and *effectively* on the space of vector lines in $\mathbb{R}^{n+1}$, *i.e.*, on $\mathbb{R}P^n$, but not *freely*.[24] Indeed, let $Stab([\eta]) \subset PGL(n+1,\mathbb{R})$ be the stabilizer of a vector line $[\eta]$, *i.e.*, the set of elements $H \in GL(n+1,\mathbb{R})$ defined up to a nonvanishing factor and for which there exists $h \in \mathbb{R}^*$ such that $H(\eta) = h\,\eta$. Then, $Stab([\eta])$ is not reduced to the singleton $\{\mathbb{1}\}$. Actually, we have an affine group, *i.e.*, $Stab([\eta]) \simeq Aff(n,\mathbb{R}) \equiv \mathbb{R}^n \rtimes GL(n,\mathbb{R})$. As a result, in term of homogeneous space, we have $\mathbb{R}P^n \simeq G/G'$ where $G \equiv PGL(n+1,\mathbb{R})$ and $G' \equiv Aff(n,\mathbb{R})$.[25]

For instance, let us consider the *homogeneous matrix* $H \in \mathbb{R}\times Stab([\zeta_0])$. Then, in the projective frame $\Psi \equiv \{[\zeta_0],\dots,[\zeta_{n+1}]\}$, we have $H(\zeta_0) = h\,\zeta_0$ and $H(\zeta_k) = \zeta'_k = H^\alpha_k\,\zeta_\alpha \equiv h\,F^\alpha_k\,\zeta_\alpha$.[26] As a result, we deduce from any vector $\eta \equiv x^\alpha\,\zeta_\alpha \in \mathbb{R}^{n+1}$ such that $\eta \notin P_1$ (*i.e.*, $x^0 \neq 0$) and $H(\eta) = \eta' = x'^\beta\,\zeta_\beta$ the following affine transformations with their corresponding well-known *conformal transformations* on the n-tuples ($i = 1,\dots,n$):

$$\left\{\begin{aligned} x'^0 &= h\,x^0 + \textstyle\sum_{k=1}^n H^0_k\,x^k \\ x'^i &= \textstyle\sum_{k=1}^n H^i_k\,x^k \end{aligned}\right\} \quad\Longrightarrow\quad \kappa'^i = \frac{\sum_{k=1}^n F^i_k\,\kappa^k}{1+\sum_{j=1}^n F^0_j\,\kappa^j}\,, \tag{3.8}$$

where we have $\kappa^i \equiv x^i/x^0$ and similar definitions for prime symbols.

Consequently, it matters to notice that whereas the projective group $PGL(n+1,\mathbb{R})$ acts on $\mathbb{R}P^n$ or on any tangent projective space $[\mathrm{T}_p\mathbb{R}^{n+1}]$ at $p \in \mathbb{R}P^n$ — homeomorphic to $\mathbb{R}P^n$ when identifying $\mathbb{R}^{n+1}$ with $\mathrm{T}_p\mathbb{R}^{n+1}$ — the structural group of the projective tangent bundle $[\mathrm{T}\mathbb{R}^{n+1}] \equiv \mathrm{T}\mathbb{R}P^n$ is, instead actually, the affine group $Aff(n,\mathbb{R}) \subset PGL(n+1,\mathbb{R})$ of elements (F^0_j, F^i_k). Indeed, at any point $p \in \mathbb{R}P^n$, the tangent projective space $\mathrm{T}_p\mathbb{R}P^n \equiv [T_p\mathbb{R}^{n+1}]$ is based actually upon the set of vectors $q - p \in \mathbb{R}^{n+1}$ where p and q are considered as points of the affine space $\mathbb{R}^{n+1}$ and, in particular, $p \in P_0$. Therefore, p is fixed as the common origin

[24] Let us note that $PGL(n+1,\mathbb{R})$ is connected if $n+1$ is odd and has two connected components otherwise. Also, sometimes, it is more convenient to work with the projective special linear group $PSL(n+1,\mathbb{R}) \simeq SL(n+1,\mathbb{R})/Z$ where Z is the center of $SL(n+1,\mathbb{R})$ and $Z = \{\mathbb{1}\}$ if $n+1$ is odd and $Z = \{\pm\mathbb{1}\}$ otherwise. Also, $PSL(n+1,\mathbb{R})$ is the identity component of $PGL(n+1,\mathbb{R})$.

[25] We recall also that $\dim PGL(n+1,\mathbb{R}) = (n+1)^2 - 1$ and $\dim Aff(n,\mathbb{R}) = n^2+n$. Thus, we obtain $\dim \mathbb{R}P^n = \dim PGL(n+1,\mathbb{R})/Aff(n,\mathbb{R}) = n$. Obviously, we have also $GL(n+1,\mathbb{R})/\left(\mathbb{R}^*\times Aff(n,\mathbb{R})\right) = \mathbb{R}P^n$.

[26] Henceforth and throughout, we adopt Einstein convention for summations and we utilize latin indices for summation from 1 to n and greek indices for summation from 0 to n.

of all of such vectors, and thus, as the origin also of the projective tangent space $[T_p\mathbb{R}^{n+1}]$ at p. Therefore, the action of $PGL(n+1,\mathbb{R})$ on $[T_p\mathbb{R}^{n+1}]$ must be such that the vector line $[\overrightarrow{op}] \subset \mathbb{R}^{n+1}$ — which is somehow the origin of the affine space $[T_p\mathbb{R}^{n+1}]$ — remains unchanged, *i.e.*, $PGL(n+1,\mathbb{R})$ must be restricted to its subgroup $Stab([\overrightarrow{op}]) \simeq Aff(n,\mathbb{R})$.

More generally, all of the changes of projective frames in $\mathrm{T}_p\mathbb{R}P^n$ must be accompanied by the invariance of a particular common vector line in $T_p\mathbb{R}^{n+1}$, and the invariance also of the representing manifold of $T_p\mathbb{R}P^n$ in $T_p\mathbb{R}^{n+1}$. We have just shown above that the former common vector line can be $[\overrightarrow{op}] \in T_p\mathbb{R}P^n$ but other point in $T_p\mathbb{R}P^n$ can be also chosen and fixed. It depends only on the choice made for the origin $s_{o,p}$ in $T_p\mathbb{R}^{n+1}$ of the representing manifold of each tangent projective space $T_p\mathbb{R}P^n$. Hence, we see that if π_p is the Yano-Ishihara projecting form on $T_p\mathbb{R}^{n+1}$ defining the representing manifold of the projective space $T_p\mathbb{R}P^n$, then the origin of this representing manifold can be such that $\overrightarrow{ps}_{o,p} \equiv \overrightarrow{op} \equiv \xi_p$ where $\xi_p \in T_p\mathbb{R}^{n+1}$ is the dual vector of π_p such that $\pi_p(\xi_p) = 1$. Then, we say also that $s_{o,p}$ or ξ_p indifferently is the origin of the representing manifold of $T_p\mathbb{R}P^n$. Moreover, we deduce the following:

Proposition 1. *At any point $p \in \mathbb{R}P^n$, all of the changes $[\mathfrak{C}] \in PGL(n+1,\mathbb{R})$ of projective frames in the tangent projective space $T_p\mathbb{R}P^n$ — from any projective frame $\mathfrak{F}_p \equiv \{[\mathfrak{v}_0],\ldots,[\mathfrak{v}_{n+1}]\}_p$ of $T_p\mathbb{R}P^n$ to any other projective frame $\mathfrak{F}'_p$ of $T_p\mathbb{R}P^n$ — and preserving the 'representing manifold' associated with the projective space $T_p\mathbb{R}P^n$ are such that $[\mathfrak{C}] \in Stab([\xi_p]) \simeq Aff(n,\mathbb{R})$. Thus, each $[\mathfrak{C}]$ preserves the origin ξ_p in $T_p\mathbb{R}^{n+1}$ of the representing manifold of $T_p\mathbb{R}P^n$ and defined by the projecting dual form π_p over $\mathbb{R}P^n$ such that $\pi_p(\xi_p) = 1$. Furthermore, given $\mathfrak{F}_p$ and $\mathfrak{F}'_p$ then $[\mathfrak{C}]$ is unique.*

In addition, the strong discrepancy between the two "*homogeneous*" Lie groups, namely, $\mathbb{R}^*\times Aff(n,\mathbb{R})$ and the linear group $GL(n+1,\mathbb{R})$ is at the origin of the different categories of projective tensors associated, somehow, with the orbits of $\mathbb{R}^*\times Aff(n,\mathbb{R})$ in $GL(n+1,\mathbb{R})$; a discrepancy between the Euclidean tensors which does not exist in vector spaces with the actions of linear groups which are always *free*. Moreover, the affine aspect of the projective tensors is revealed in the Grassmann algebra of $\mathbb{R}P^n$ by *filtrations* — associated with the affine aspect of the projective geometry — rather than by *graduations* like on the Grassmann algebras of Euclidean tensors again. In other words, this discrepancy is due to the non-equivalence between the contragredient actions and the dual actions of $\mathit{Aff}(n,\mathbb{R})$ on cotensors or, equivalently, the actions on tensors and the actions on cotensors of $\mathit{Aff}(n,\mathbb{R})$ are not dual, contrary to the linear groups actions on Euclidean (co-)tensors.

Now, we present and recall what are the general projective Cartan connections [Car24b, CS07, Arm08a, Arm08b, Cra09] and we finish with a little outline on the projective (co-)tensors in a way developed historically by É. Cartan and which is certainly one of the most concise and clear existing presentation. Moreover, it does not exist english translation of his seminal papers on projective connections and projective tensors and thus this is also an opportunity to present his own initial formalism on this differential geometry.

C. The projective Cartan connections

Important remark: *We use the following terminology: we call* 'projective connection' *the most general projective connection. The latter corresponds to the* '(projective) [Cartan] connection' *of type F over B defined by Ehresmann (see p. 23). We use rather the terminology* 'Ehresmann connections' *for these connections defined by Ehresmann within the context of his definition of generalized Cartan spaces. Then, we keep the terminology* 'projective Cartan connections' *for those particular projective connections determined by Cartan (in his 1924 seminal paper) and which correspond, somehow, to normal forms for projective connections as Jordan forms correspond similarly to normal forms for square matrices.*

1. *The general case – Definitions and terminologies*

We consider the first following situation. Let $\mathbb{R}P^n \simeq P_0 \cup \mathbb{R}P^{n-1}$ be a projective space with the given and fixed projective frame $\mathfrak{F}_0 \equiv \{[\zeta_0], \ldots, [\zeta_{n+1}]\}$ and constant projective form π_0 (with its dual vector ξ_0). Each point $[\eta] \in \mathbb{R}P^n$ can be locally represented in P_0, via a suitable adapted local homeomorphism, by a point $p \in P_0$ with the system of local inhomogeneous coordinates $[\kappa^1, \ldots, \kappa^n]_n$ defined from $\mathfrak{F}_0$. Also, we consider that each tangent space $T_p\mathbb{R}P^n \simeq \mathfrak{P}_0(p) \cup T_p\mathbb{R}P^{n-1}$ at any $p \in P_0$ is a projective space endowed itself with the following objects:

- a Yano-Ishihara '*projecting 1-form*' $\pi(p)$ in $T_p^*\mathbb{R}^{n+1}$ with no singularities in $\mathbb{R}P^n$ and with its given dual vector $\xi(p) \in T_p\mathbb{R}^{n+1}$ such that $\pi(\xi)(p) = 1$,
- a '*representing (affine) manifold*' $\mathfrak{P}_0(p) \subset T_p\mathbb{R}^{n+1}$ of dimension n with origin s_p such that $\overrightarrow{ps_p} = \xi$ and its corresponding parallel n-dimensional vector space $\mathfrak{P}_1(p)$,
- a projective frame $\mathfrak{F}(p) = \{[\mathfrak{v}_0](p), \ldots, [\mathfrak{v}_{n+1}](p)\}$ such that everywhere on P_0 we have $\pi(\mathfrak{v}_i) = 0$ for all $i = 1, \ldots, n$ and $\pi(\mathfrak{v}_0)\pi(\mathfrak{v}_{n+1}) \neq 0$,
- an Euclidean basis $\mathfrak{B}(p) = \{\mathfrak{v}_0(p), \ldots, \mathfrak{v}_n(p)\}$ of $T_p\mathbb{R}^{n+1}$ and its dual Euclidean cobasis $\mathfrak{B}^*(p) = \{\mathfrak{v}^{*,0}(p), \ldots, \mathfrak{v}^{*,n}(p)\}$ such that $\mathfrak{v}^{*,\alpha}(\mathfrak{v}_\beta) = \delta^\alpha_\beta$,
- the free sub-system of generators $\mathcal{B}(p) = \{\mathfrak{v}_1(p), \ldots, \mathfrak{v}_n(p)\}$ of $\mathfrak{P}_1(p)$ associated with $\mathfrak{B}(p)$ and its dual system $\mathcal{B}^*(p) = \{\mathfrak{v}^{*,1}(p), \ldots, \mathfrak{v}^{*,n}(p)\}$ such that $\mathfrak{v}^{*,i}(\mathfrak{v}_j) = \delta^i_j$,
- two supplementary projectors $\mathfrak{q}_0(p)$ and $\mathfrak{p}_0(p)$; the latter projecting on the vector space $\mathfrak{P}_1(p)$ parallel to the affine space $\mathfrak{P}_0(p)$.

We consider these geometrical objects to be the values at p of the fields $\mathfrak{B}$, $\mathcal{B}$, $\mathfrak{P}_0$, $\mathfrak{F}$, π, ξ, $\mathfrak{p}_0$ and $\mathfrak{q}_0$ on $\mathbb{R}P^n$ (*i.e.*, locally *soldered* to P_0) and thus depending locally on the inhomogeneous coordinates κ^i.

Now, we generalize this first situation. Actually, we generalize the notion of representing manifold, and first, we present some criteria for the existence of local trivializations of a manifold M.

a. ***The foliations*** — For, we consider a *paracompact connected* manifold M of dimension $n+1$ and class C^r $(r \geqslant 1)$, and a 1-form $\pi \in T^*M$ of class C^{r-1} with no singularities in M, *i.e.*, *regular*. Moreover, if $r \geqslant 2$ then we can assume that π is *integrable*, *i.e.*, we have $d\pi \wedge \pi = 0$ or, equivalently, from the Frobenius conditions: $d\pi = \alpha \wedge \pi$ where α is another 1-form defined on M modulo π.

As a result, from the Frobenius theorem, π defines a codimension one foliation $\mathcal{F}$ of class C^{r-1} on M (see [God91, p.6] and footnote 11, p. 13) of maximal integrable *connected* leaves $\mathcal{J}$ of dimension n and class C^{r-1}. Also, due to the paracompacity of M, $\mathcal{F}$ is *transversally orientable*, *i.e.*, the normal bundle $\mathcal{V}(\mathcal{F}) \subset TM$ over M with fibers of dimension one is orientable [God91, p.5]. Moreover, because we have a codimension one differentiable foliation, there always exists a transversal vector field ξ on M, *i.e.*, a vector field such that $\pi(\xi) \neq 0$. And therefore, there always exists a foliation $\mathcal{T}$ of dimension one and of class C^{r-1} which is transversal to $\mathcal{F}$ [God91, Remarks 2.15, p.18]). In addition, if we assume that the leaves $\mathcal{J}$ are *non-closed* sets (*e.g.*, P_0 or a half of a non closed equator joined to an open hemisphere) and because M is transversally orientable, then, each leaf of $\mathcal{T}$ is a *closed* submanifold of M (see [God91, Proposition 2.17, p.18] or [God83]); and therefore, diffeomorphic to a closed interval of $\mathbb{R}$ or S^1.

Then, we choose a particular leaf $\mathcal{J}_0 \in \mathcal{F}$, and we can show that each leaf $\mathcal{J}$ of $\mathcal{F}$ can be *modeled* on $\mathbb{R}P^n$ as the following.

Indeed, as a result, we can find also a *surmersion* $W : M \longrightarrow \mathcal{J}_0$ of class C^{r-1} of which the horizontal sets are the leaves $\mathcal{J}$ *projectable* (*i.e.*, transversal to W) onto $\mathcal{J}_0$ and such that M is locally trivializable, *i.e.*, M is a fiber bundle.

Then, given local charts for M and for $\mathcal{J}_0$ with the topology induced by M, we consider that we have *homogeneous coordinates* on M of which the corresponding *inhomogeneous coordinates* are those defined by the charts given on $\mathcal{J}_0$. In other words, $\mathcal{J}_0$ is locally a representing manifold of $\mathbb{R}P^n$ as P_0, *i.e.*, $\mathcal{J}_0$ is *modeled* on $\mathbb{R}P^n$, and we write $\mathcal{J}_0 \simeq_{loc.} \mathbb{R}P^n$. Obviously, this is also the case for any leaf $\mathcal{J}$.

We can note that $M = \mathcal{U} \times K$ where $K = \mathbb{R}$ or S^1, in particular cases only; For instance, if there exist a complete vector field ξ and a closed regular 1-form π in a connected, paracompact, without boundaries and simply connected manifold M of class $C^{r \geqslant 1}$ [God91, see §4, pp.45–50 for details and references therein]. Also, in this case, M has no holonomy [God91, p.94, vi)]. In particular, if $K = \mathbb{R}$ then M is necessarily non closed from the Tischler's theorem [Tis70].

Moreover, if there exists a *complete* vector field ξ of class C^{r-1} on M ($r \geqslant 2$) we deduce that the closed transversal leaves of $\mathcal{T}$ are diffeomorphic to $\mathbb{R}$ or S^1 (Also, in full generality, we can note that the flow $\varphi(t,p)$ of ξ does not preserve the foliation $\mathcal{F}$, *i.e.*, $\varphi(t,\mathcal{J})$ ($t \neq 0$) is not always included in a leaf).

Then, to summarize, we assume that

1. *M is a paracompact connected manifold of dimension $n+1$ and of class C^r with $r \geqslant 2$,*

2. *π is a regular integrable 1-form of class C^{r-1} on M,*

3. *the leaves $\mathcal{J}$ of the foliation $\mathcal{F}$ which are the maximal, integral, n-dimensional and connected manifolds of class C^{r-1} defined by π are assumed to be non-closed, and*

4. *there exists a complete vector field ξ of class C^{r-1} on M such that $\pi(\xi) = 1$.*

b. ***The distinguished diffeomorphisms*** — Additionally, we set the following definitions.

Definition 1. *We call the 1-form π the* 'Yano-Ishihara 1-form*' or the* 'projecting form*' of the projective structure on M [YI67, denoted by $\tilde{\eta}$].*

Also, from W, we define *distinguished diffeomorphisms* as in the paragraphs above as following .

Definition 2. *We call* 'projective distinguished diffeomorphism*' on M any local diffeomorphism f of class C^r $(1 \leqslant r \leqslant \infty)$ on M from any open $\mathcal{U} \subset M$ to $\mathcal{U}' \equiv f(\mathcal{U})$ such that*

1. *there exists a C^{r-1} function a on $\mathcal{U}$ such that $f^*(\pi) = a\,\pi$, and*

2. *f is a (fiber preserving) 'bundle map' of M over $\mathcal{J}_0$* covering *a projective (transformation) map $\wp$ defined on $\mathcal{J}_0 \simeq_{loc.} \mathbb{R}P^n$ by an element of $PGL(n+1, \mathbb{R})$, i.e., we have the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{f} & \mathcal{U}' \\ {\scriptstyle W}\big\downarrow & & \big\downarrow{\scriptstyle W} \\ W(\mathcal{U}) \cap \mathcal{J}_0 & \xrightarrow{\wp} & W(\mathcal{U}') \cap \mathcal{J}_0 \end{array} \tag{3.9}$$

We can say also that f is a 'leafwise conformal diffeomorphism*' and we denote by $PDiff^r_{loc.}(M, \mathcal{F}, \mathcal{J}_0) \subset Diff^r_{loc.}(M, \mathcal{F}) \subset Diff^r_{loc.}(M)$ the pseudogroup of such projective local distinguished diffeomorphisms ($Diff^r_{loc.}(M, \mathcal{F})$ is the pseudogroup of local distinguished diffeomorphisms of class C^r*

preserving the foliation $\mathcal{F}$, i.e., the set of diffeomorphisms such that only the condition 1 above is satisfied).

Remark 1. *We must notice that* global *projective distinguished diffeomorphisms cannot exist on M because the projective maps $\wp$ are singular on closed hyperplanes of dimension n, and thus, homeomorphic to projective varieties of dimension $n-1$.*

Remark 2. *Actually, to find Ehresmann connections is a problem reduced to find sets of scale invariant homogeneous functions of degree zero in the case of projective connections (relative to the Eulers' theorem on homogeneous functions). Then, from these invariant functions (constituting a closed differential ideal with n generators) we can deduce maps from the systems of homogeneous coordinates to the systems of inhomogeneous coordinates related to the submersions W. But, this is completely related to the existence of $Ad(\mathcal{G})$-invariant integrable systems of Pfaff 1-forms which define the systems $\widehat{C}$ of integral elements in the principal frame bundles $P(B,G)$ defined by Ehresmann. From the dual vector fields of these Pfaff 1-forms on $P(B,G)$ (with charts of homogeneous coordinates), we can deduce the invariant functions and then, by reduction with the homogenous invariant functions, the corresponding integrable systems C of Pfaff 1-forms on the vector bundle $E(B,F,G)$ (with charts of inhomogeneous coordinates). As a result, we obtain a unified way linking the three historical approaches:*

- Ehresmann*: the systems $\widehat{C}$ of integral elements in the principal frame bundles $P(B,G)$ followed by the reduction process to C,*

- Cartan*: the system of ordinary differential equations defining the infinitesimal changes of (projective) frames [Car24b, see, for instance, system of ODEs (3')] with trivial, implicit and somehow hidden sets of homogeneous functions of degree zero,*

- Veblen and al. and Schouten and *al.: the systems of inhomogeneous functions and their subsequent scale invariances with hidden or not explicit sets of projective frames with their infinitesimal changes expressed in terms of explicit systems of ODEs.*

Hence, all can be reduced to find the systems $\widehat{C}$ with their invariant functions, and then, their reduced systems C. This can be done in other situations than those met with projective manifolds and, in particular, in the cases of Grassmaniann manifolds. But, the explicit determinations of such systems $\widehat{C}$ has been done completely and classified by P. Winternitz and al. *in his papers on the so-called '*systems of ordinary differential equations with non-linear principles of superposition*' [AHW81b, AHW81a, AHW82, Win82, HWA83, SW84b, SW84a, Win84, SW85, BHW86, BPW86a, BPW86b, DRW86, BHW87, GHW88, HPW99, TW99, HPW01]. Nevertheless, we can note that the Grassmaniann connections have been determined first by S.S. Chern [Che43, Che45].*

Remark 3. *The condition 2 provides an analog of the fiber isomorphisms in the condition (c) of Ehresmann. Indeed, we have just to consider that 1) M is the analog of $E(B,F,G)$, 2) $\mathcal{J}_0$ is the analog of F, 3) each leaf $\mathcal{J}$ is an analog of B, and the field of integral elements of π is the analog of C. Moreover, given local sections $s_{W,\mathcal{U}} : W(\mathcal{U}) \cap \mathcal{J}_0 \longrightarrow \mathcal{U} \cap \mathcal{J}$ of the submersion W provides an analog of the* soldering *diffeomorphism S_g defined by Ehresmann ($g \in G/G'$).*

Moreover, from these local sections $s_{W,\mathcal{U}}$, we can obtain diffeomorphisms preserving any given leaf $\mathcal{J}$ from diffeomorphisms preserving only the foliation, i.e., from distinguished diffeomorphisms.

Also, the sections of W can be the analogs of the homeomorphisms h_z, but they cannot be parameterized in full generality by (are homeomorphic to) $n+1$ vector fields (or $n+2$ vector lines in the tangent spaces) defined, up to a scalar function, on each $W(\mathcal{U}) \cap \mathcal{J}_0$ and constituting a projective frame field $\mathfrak{F}$. Indeed, the codimensions of the bundles differ.

Thenceforth, we can say that the manifold $\mathcal{J}_0$ is the representing manifold of $\mathbb{R}P^n$. No projective frames are defined on $\mathcal{J}_0$ as they are on P_0 from vector lines in $\mathbb{R}^{n+1}$. Actually, this is replaced by the condition 2 in Definition 2 and also because $\mathcal{J}_0$ inherits its systems of local "inhomogeneous" coordinates $(\kappa^1, \ldots, \kappa^n)$ from the surmersions W via local charts of "homogeneous" coordinates on $\mathcal{U}$ to $\mathbb{R}^{n+1}$ and via local charts of "inhomogeneous" coordinates on $W(\mathcal{U}) \cap \mathcal{J}_0$ to the affine space $P_0 \subset \mathbb{R}^{n+1}$. The surmersions W can be merely projections for instance, but also homogeneous maps of degree zero (respectively to the Euler's theorem for homogeneous functions). In fact, this last viewpoint is the one presented historically by authors such as J. A. Schouten; in particular, explicitly in the section of introduction of his 1935 paper [Sch35, in french].

Then, we have the following supplementary geometric objects:

- a '*projecting (Yano-Ishihara) 1-form*' π in T^*M with no singularities in M and with its given dual complete vector field $\xi \in \chi(M)$ such that $\pi(\xi) = 1$,
- a '*representing (affine) manifold*' bundle of representing manifolds $\mathfrak{P}_0(p) \subset T_pM$ of dimension n with origins $s_p \in T_pM$ such that $\overrightarrow{ps_p} = \xi(p)$ and its corresponding parallel n-dimensional vectorial bundle of parallel n-dimensional vector spaces $\mathfrak{P}_1(p) \subset T_pM$ which is the reduced bundle from TM defined by the set of all integral elements of dimension n annihilated by π (in other words, we have $\mathfrak{P}_1(p) \equiv T_p\mathcal{J}$ for each $p \in \mathcal{J}$),
- a principal bundle $\mathfrak{F}_{GL(n+1,\mathbb{R})}(TM)$ of projective frames $\mathfrak{F}(p) = \{[\mathfrak{v}_0](p), \ldots, [\mathfrak{v}_{n+1}](p)\}$ over $p \in M$ and with structural group $PGL(n+1, \mathbb{R})$, and thus, such that everywhere on M we have $\pi(\mathfrak{v}_i) = 0$ for all $i = 1, \ldots, n$ and $\pi(\mathfrak{v}_0)\pi(\mathfrak{v}_{n+1}) \neq 0$,
- a principal bundle $\mathfrak{B}_{GL(n+1,\mathbb{R})}(TM)$ over M with structural group $GL(n+1, \mathbb{R})$ of Euclidean bases $\mathfrak{B}(p) = \{\mathfrak{v}_0(p), \ldots, \mathfrak{v}_n(p)\}$ of T_pM and its dual principal bundle of dual Euclidean cobases $\mathfrak{B}^*(p) = \{\mathfrak{v}^{*,0}(p), \ldots, \mathfrak{v}^{*,n}(p)\}$ such that $\mathfrak{v}^{*,\alpha}(\mathfrak{v}_\beta) = \delta^\alpha_\beta$,
- a principal bundle $\mathcal{B}_{GL(n,\mathbb{R})}(TM)$ over M with structural group $GL(n, \mathbb{R})$ of free sub-systems of generators $\mathcal{B} = \{\mathfrak{v}_1, \ldots, \mathfrak{v}_n\}$ of the vector spaces $\mathfrak{P}_1$ associated with the bases $\mathfrak{B}$ and their dual systems $\mathcal{B}^* = \{\mathfrak{v}^{*,1}, \ldots, \mathfrak{v}^{*,n}\}$ such that $\mathfrak{v}^{*,i}(\mathfrak{v}_j) = \delta^i_j$,
- two fields of supplementary projectors $\mathfrak{q}_0$ and $\mathfrak{p}_0$; the latter projecting on the vector spaces $\mathfrak{P}_1$ "parallel" to the affine spaces $\mathfrak{P}_0$.

c. ***The inhomogeneity and the horizontal/vertical splitting*** — Besides, we have the following definitions.

Definition 3. *We call r-forms or (co-)tensor fields defined on spaces homomorphic to Euclidean spaces $\mathbb{R}^{n+1}$ but depending only, as fields, on the* inhomogeneous coordinates *taken for the points in $\mathbb{R}P^n$, '*inhomogeneous*' r-forms or (co-)tensors fields, or, also, r-forms or (co-)tensors fields "*over*" $\mathbb{R}P^n$. On the contrary, '*homogeneous*' forms are defined on the Euclidean space $\mathbb{R}^{n+1}$ and depend fully on the homogeneous coordinates.*

Then, relative to any submersion W, we give the following coordinates independent definition and generalization of inhomogeneity.

Definition 4. *We call "*inhomogeneous*" tensors on $\mathcal{U} \subset M$ any tensor of the tensor algebra $\mathfrak{T}(TM)$ which is the push-forward by a section of a submersion $W : \mathcal{U} \longrightarrow W(\mathcal{U}) \cap \mathcal{J}_0$ of a tensor of $\mathfrak{T}(TM)$ defined on $W(\mathcal{U}) \cap \mathcal{J}_0$.*

Definition 5. *We call "*inhomogeneous*" cotensors (resp. r-forms) on $\mathcal{U} \subset M$ any cotensor (resp. r-form) of the cotensor algebra $\mathfrak{T}(T^*M)$ (resp. the exterior algebra $\wedge T^*M$) which is the pull-back by a submersion $W : \mathcal{U} \longrightarrow W(\mathcal{U}) \cap \mathcal{J}_0$ of a cotensor of the cotensor algebra $\mathfrak{T}(T^*M)$ (resp. r-form of $\wedge T^*M$) defined on $W(\mathcal{U}) \cap \mathcal{J}_0$.*

From these definitions, for instance, a vector can be horizontal but non-inhomogeneous if its components depend on all of the $n+1$ local coordinates defined on $\mathcal{U}$. Moreover, we have also:

Definition 6. *Any q-form ϕ (or cotensor) such that $i_\xi \phi = 0$ where i_ξ is the interior product associated with ξ is said '*horizontal*' q-form (cotensor) ϕ. Any tensor Σ annihilated by π, i.e., such that $i_\Sigma \pi = 0$ is call '*horizontal*' tensor.*

Then, a vector field $\mathfrak{v}$ in $\chi(\mathbb{R}P^n)$ is a vector field in $\chi(M)$ which is inhomogeneous and horizontal. Also, in particular, the projective connections are horizontal 1-forms T^*M *over* $\mathbb{R}P^n$ (and thus horizontal and inhomogeneous). On the contrary, the Yano-Ishihara projecting forms π are '*vertical*' 1-forms over $\mathbb{R}P^n$, and then, any r-form ψ over $\mathbb{R}P^n$ such that $\pi \wedge \psi \equiv 0$

is also said '*vertical*.' Somehow, horizontality over $\mathbb{R}P^n$ is the criteria to speak of forms or (co-)tensors "*on*" $\mathbb{R}P^n$.

Remark 4. *At this point, we ought to draw attention also to a source of confusion in the utilizations of the words* "horizontal" *and* "vertical" *depending on which foliation we consider.*

Indeed, the projective space $\mathbb{R}P^n$ is, in the present situation, the standard fiber F of the Ehresmann's bundles $E(B, F, G)$, and therefore, we could say also that the manifolds $\mathcal{J}$ are vertical too. Actually, this horizontal/vertical splitting refers to the foliation of $E(B, F, G)$ foliated by a system C of contact elements (i.e., Pfaff forms). But this is not the present and subsequent geometrical framework we use and which is featured by the foliation determined by π.

Then, from now and throughout, unless otherwise, explicitly stated exceptions, we consider that the horizontal/vertical splitting refers always to the foliation defined by π, and then, the integral leaves $\mathcal{J}$ as well as $\mathbb{R}P^n$ are considered both as horizontal *manifolds.*

d. ***The connections*** — Thenceforth, to define a (general) projective connection on $\mathbb{R}P^n$ we consider first, for the sake of argument, the simpler case where $M \equiv \mathbb{R}^{n+1}$. Then, we denote by $\mathfrak{F}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$ the principal bundle of projective frames $\mathfrak{F}$ over M, and we take the foliation $\mathcal{F}$ such that $\mathcal{F} \equiv \cup_{t=-\infty}^{+\infty} \mathbb{R}^n \times \{t\}$ and $P_0 \equiv \mathcal{J}_0 = \mathbb{R}^n \times \{1\}$. It follows that $\pi_0 \equiv \pi$ and locally we have $P_0 \simeq_{loc.} \mathbb{R}P^n$. Then, if $p \in P_0$ we can use indifferently p or $[p]$ for an element of $\mathbb{R}P^n$, *i.e.*, $p \simeq [p]$. Nevertheless, in full generality, we take $\xi \neq \xi_0$.

Moreover, we recall also the following notions on the connections in the smooth category. First, we denote by $End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$ the bundle of endomorphisms of $T\mathbb{R}^{n+1}$ over $\mathbb{R}^{n+1}$ with structural group $GL(n+1, \mathbb{R})$ and with standard fiber the monoid $End(T\mathbb{R}^{n+1})$.

Then, let $\mathcal{G} : M \longrightarrow End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}) \otimes T^*\mathbb{R}^{n+1}$ be a smooth section of the fiber bundle $End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}) \otimes T^*\mathbb{R}^{n+1}$, *i.e.*, a square matrix-valued 1-form on M. We have in particular $\mathfrak{B}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}) \subset End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$ and, moreover, $End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$ can be understood as the fiber bundle of which the elements of the fibers are $(n+1)$-square matrices not necessarily invertible contrary to the elements of the fibers of $\mathfrak{B}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$.

Hence, we have for $p \in M$ that $\mathcal{G}(p) = \{\mathfrak{w}_0(p), \ldots, \mathfrak{w}_n(p)\} \in End_{GL(n+1,\mathbb{R})}(T_p\mathbb{R}^{n+1}) \otimes T_p^*\mathbb{R}^{n+1}$ where the $\mathfrak{w}$'s are smooth vector-valued 1-forms on M not necessarily all linearly independent. Then, from $\mathcal{G}$ we can defined a map $\omega : \mathfrak{B}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}) \longrightarrow C^\infty(\mathbb{R}^{n+1}, End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}) \otimes T^*\mathbb{R}^{n+1}) \equiv \Gamma_{n+1}(End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}) \otimes T^*\mathbb{R}^{n+1})$ such that for all smooth basis field $\mathfrak{B} \equiv \{\mathfrak{v}_0, \ldots, \mathfrak{v}_n\} \in \mathfrak{B}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$ we have

$$\omega(\mathfrak{B})(p) \equiv (\mathcal{G}^\alpha_\beta(p)) \in End_{GL(n+1,\mathbb{R})}(T_p\mathbb{R}^{n+1}) \otimes T^*_p\mathbb{R}^{n+1} \tag{3.10}$$

where

$$\mathcal{G}^\alpha_\beta \equiv \mathfrak{v}^{*,\alpha}(\mathcal{G} \,.\, \mathfrak{v}_\beta)\,. \tag{3.11}$$

Thus, we have also $\mathcal{G}(p) = \sum_{\alpha,\beta=0}^n \mathcal{G}^\alpha_\beta(p)\ \mathfrak{v}_\alpha(p) \otimes \mathfrak{v}^{*,\beta}(p)$. In addition, we use also the notation $\omega(\mathfrak{B}) \equiv \omega_{\mathfrak{B}} = (\omega^\alpha_{\mathfrak{B},\beta})$ and thus $\mathcal{G}^\alpha_\beta = \omega^\alpha_{\mathfrak{B},\beta}$. Hence, $\omega_{\mathfrak{B}}$ is a matrix in the canonical basis $\{\mathfrak{i}_0, \ldots, \mathfrak{i}_n\}$ of $\mathbb{R}^{n+1}$ which is a basis *independent on* M. Thus, we have $\omega_{\mathfrak{B}}(p) = \sum_{\alpha,\beta=0}^n \omega^\alpha_{\mathfrak{B},\beta}(p)\ \mathfrak{i}_\alpha \otimes \mathfrak{i}^{*,\beta}$.

Then, the matrix $\mathcal{G}$ is represented in a basis which is, somehow, "abstract" respective to the bases given on the base space $M \equiv \mathbb{R}^{n+1}$, *i.e.*, the matrix-valued 1-form $(\mathcal{G}^\alpha_\beta)$ contracted by a vector field is an element of a monoid (a fiber) in the bundle $End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$.

Then, if $\mathfrak{H}$ is a Euclidean basis field on M, *i.e.*, a smooth map from M to $\mathfrak{B}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$, we define the right action $R_{\mathfrak{H}}$ of $\mathfrak{H}$ on $\mathfrak{B}$ such that $R_{\mathfrak{H}}\mathfrak{B} = \mathfrak{B}' = \{\mathfrak{H}\,\mathfrak{v}_0, \ldots, \mathfrak{H}\,\mathfrak{v}_n\}$ where, by convention, $\mathfrak{H}\,\mathfrak{v}_\alpha = \mathfrak{v}'_\alpha = \sum_{\beta=0}^n \mathfrak{H}^\beta_\alpha\,\mathfrak{v}_\beta$, and thus, $\mathfrak{H}^\beta_{\mathfrak{B},\alpha} \equiv \mathfrak{H}^\beta_\alpha = \mathfrak{v}^{*,\beta}(\mathfrak{H} \,.\, \mathfrak{v}_\alpha)$. Henceforth, we can also define the right action $R^*_{\mathfrak{H}}$ of $R_{\mathfrak{H}}$ on ω:

$$(R^*_{\mathfrak{H}}(\omega)(\mathfrak{B}))^\alpha_\beta \equiv (\omega(R_{\mathfrak{H}}\mathfrak{B}))^\alpha_\beta = \mathfrak{v}'^{*,\alpha}(\mathcal{G} \,.\, \mathfrak{v}'_\beta)\,. \tag{3.12}$$

And thus, we obtain that

$$
\begin{aligned}
(R^*_{\mathfrak{H}}(\omega)_{\mathfrak{B}})^\alpha_\beta &= \mathfrak{v}'^{*,\alpha}(\mathcal{G}\,.\,\mathfrak{v}'_\beta) \\
&= \sum_{\gamma=0}^{n} \mathfrak{H}^\gamma_\beta\, \mathfrak{v}'^{*,\alpha}(\mathcal{G}\,.\,\mathfrak{v}_\gamma) \\
&= \sum_{\gamma,\mu=0}^{n} \mathfrak{H}^\gamma_\beta\, \mathfrak{H}^{-1,\alpha}_\mu\, \mathfrak{v}^{*,\mu}(\mathcal{G}\,.\,\mathfrak{v}_\gamma) \\
&= \sum_{\gamma,\mu=0}^{n} \mathfrak{H}^\gamma_\beta\, \mathfrak{H}^{-1,\alpha}_\mu\, \omega^\mu_{\mathfrak{B},\gamma} \\
&= \mathfrak{H}^{-1}_{\mathfrak{B}}\,.\,\omega_{\mathfrak{B}}\,.\,\mathfrak{H}_{\mathfrak{B}} && \text{(3.13a)} \\
&\equiv Ad(\mathfrak{H}^{-1}_{\mathfrak{B}})\,\omega_{\mathfrak{B}} && \text{(3.13b)}
\end{aligned}
$$

where $\mathfrak{H}_{\mathfrak{B}} \equiv (\mathfrak{v}^{*,\alpha}(\mathfrak{H}\,.\,\mathfrak{v}_\beta))$ and where $Ad(\mathfrak{C})\,\mathfrak{A} \equiv \mathfrak{C}\,.\,\mathfrak{A}\,.\,\mathfrak{C}^{-1}$ is the '*adjoint action*' of $\mathfrak{C}$ on $\mathfrak{A}$. We have also[27]

$$(\mathfrak{H}\,\mathfrak{A})_{\mathfrak{B}} = \mathfrak{H}_{\mathfrak{B}}\,\mathfrak{A}_{\mathfrak{B}}\,, \tag{3.14a}$$

$$R_{\mathfrak{H}\,\mathfrak{A}} = R_{\mathfrak{A}}\,R_{\mathfrak{H}}\,, \tag{3.14b}$$

$$R^*_{\mathfrak{H}\,\mathfrak{A}} = R^*_{\mathfrak{A}}\,R^*_{\mathfrak{H}}\,, \tag{3.14c}$$

$$Ad(\mathfrak{H}_{\mathfrak{B}}\,\mathfrak{A}_{\mathfrak{B}}) = Ad(\mathfrak{H}_{\mathfrak{B}}) \circ Ad(\mathfrak{A}_{\mathfrak{B}})\,. \tag{3.14d}$$

Now, from (3.13a) and (3.14a), we deduce that $R^*_{\mathfrak{H}}(\omega)_{\mathfrak{B}} = \big(\mathfrak{H}^{-1}.\,\omega\,.\,\mathfrak{H}\big)_{\mathfrak{B}}$ and thus, we obtain that

$$R^*_{\mathfrak{H}}(\omega) = Ad(\mathfrak{H}^{-1})\,\omega\,. \tag{3.15}$$

We must notice that the relation

$$\omega_{R_{\mathfrak{H}}\mathfrak{B}} = Ad(\mathfrak{H}^{-1}_{\mathfrak{B}})\,\omega_{\mathfrak{B}} \tag{3.16}$$

we obtain is *very general and always satisfied* for all $\mathfrak{H}$ and all $\mathfrak{B}$. This relation expresses merely the change of representation matrix of the field $\mathcal{G}$ given at a fixed point $p \in M$. In other words, (3.16) is a punctual relation.

[27] We take the following convention: for any endomorphism A then its square matrix representation $A_{\mathfrak{B}} \equiv (A^\beta_\alpha)$ in a basis $\mathfrak{B} \equiv \{\mathfrak{v}_0, \ldots, \mathfrak{v}_n\}$ is such that the coefficient A^β_α is in the column β and in the raw α. Then, we have the following important result: $(AB)^\beta_\alpha \equiv \sum_{\mu=0}^n B^\mu_\alpha\, A^\beta_\mu$, *i.e.*, $(AB)_{\mathfrak{B}} = A_{\mathfrak{B}}\, B_{\mathfrak{B}}$. Indeed, We have $\mathfrak{v}'_\alpha \equiv AB.\mathfrak{v}_\alpha = \sum_{\beta=0}^n (AB)^\beta_\alpha \mathfrak{v}_\beta$. But, moreover, we have also $\mathfrak{v}'_\alpha \equiv A\sum_{\mu=0}^n B^\mu_\alpha \mathfrak{v}_\mu = \sum_{\mu=0}^n B^\mu_\alpha A\mathfrak{v}_\mu = \sum_{\beta,\mu=0}^n B^\mu_\alpha A^\beta_\mu \mathfrak{v}_\beta$.

Now, if f is a diffeomorphism of M, then it induces a bundle isomorphism on TM and T^*M as well. As a result, we can also define the following right action of f on ω:

$$R^*_f(\omega)(\mathfrak{B}) \equiv \langle\, \omega(f_*(\mathfrak{B}))\,|\, f_*(\,.\,)\,\rangle = \langle\, \omega(R_{Tf}\mathfrak{B})\,|\, Tf(\,.\,)\,\rangle \circ f^{-1} = \langle\, (f^*\omega)(R_{Tf}\mathfrak{B})\,|\,(\,.\,)\,\rangle\,. \quad (3.17)$$

Besides, we have also the other defining relations

$$T_f f^{-1}\langle\, \omega(\mathfrak{B})\,|\,(\,.\,)\,\rangle\, Tf \equiv Ad(Tf^{-1})\langle\, \omega(\mathfrak{B})\,|\,(\,.\,)\,\rangle \equiv Ad(Tf^{-1})\,\omega(\mathfrak{B})\,. \quad (3.18)$$

But, contrary to the precedent case, *we have not always the equality*

$$\langle\, \omega(f_*(\mathfrak{B}))\,|\, f_*(\,.\,)\,\rangle = Ad(Tf^{-1})\,\omega(\mathfrak{B})\,. \quad (3.19)$$

Indeed, the field $\mathcal{G}$ is evaluated at $f(p)$ on the l.h.s. of this equality and at p on the r.h.s. contrary to the relation (3.16) where $\mathcal{G}$ is evaluated at the same point p on the two sides. And thus, given a set of diffeomorphisms f for instance and setting the equality

$$R^*_f(\omega)(\mathfrak{B}) = Ad(Tf^{-1})\,\omega(\mathfrak{B}) \quad (3.20)$$

for all $\mathfrak{B}$, then we discriminate among the maps ω those possibly suitable to satisfy this relation. In particular, given a pseudogroup of diffeomorphisms f, the maps ω satisfying (3.20) are said *right equivariant* with respect to this pseudogroup.

In addition, let $\mathfrak{F}^A = \{[\mathfrak{v}^A_0],\ldots,[\mathfrak{v}^A_{n+1}]\}$ and $\mathfrak{F}^B = \{[\mathfrak{v}^B_0],\ldots,[\mathfrak{v}^B_{n+1}]\}$ be two projective frames, *i.e.*, two classes of Euclidean bases $[\mathfrak{B}^A] \simeq \mathfrak{F}^A$ and $[\mathfrak{B}^B] \simeq \mathfrak{F}^B$ constituted by Euclidean bases $\mathfrak{B}$ differing only by scaling factors. Then, we define the right action $R_{\mathfrak{F}^A}\mathfrak{F}^B$ of $\mathfrak{F}^A$ on $\mathfrak{F}^B$ by the relation

$$R_{\mathfrak{F}^A}\mathfrak{F}^B = \{[\mathfrak{B}^A\mathfrak{v}^B_0],\ldots,[\mathfrak{B}^A\mathfrak{v}^B_{n+1}]\} \equiv \mathfrak{F}^B * \mathfrak{F}^A\,, \quad (3.21)$$

whatever is $\mathfrak{B}^A \in \mathfrak{F}^A$. In this definition, we consider that the basis fields $\mathfrak{B}$ are Euclidean basis fields on M, *i.e.*, smooth maps from M to $\mathfrak{B}_{GL(n+1,\mathbb{R})}(TM)$, and then, we can define their right action $R_{\mathfrak{B}}$ and thus we can also define $Ad(\mathfrak{B})$. Hence, from this right action, given any $p \in M$ we deduce that the manifold of projective frames $\{\mathfrak{F}(p)\}$ in T_pM is homeomorphic to the manifold $PGL(n+1,\mathbb{R})$, *i.e.*, $PGL(n+1,\mathbb{R}) \simeq \{\mathfrak{F}(p)\}$. In other words, we identify each fiber of the projective frame bundle with its structural Lie group manifold which is also its standard fiber.

Now, we are ready to define a projective connection in this quite simple case where $M = \mathbb{R}^{n+1}$, $\mathcal{F} \equiv \cup_{t=-\infty}^{+\infty} \mathbb{R}^n \times \{t\}$ and $P_0 = \mathbb{R}^n \times \{1\} \simeq_{loc.} \mathbb{R}P^n$.

Definition 7. *A classe C^r map $\omega : \Gamma_{n+1}(\mathfrak{F}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})) \longrightarrow \Gamma_{n+1}(End_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}) \otimes T^*\mathbb{R}^{n+1})$ is a* 'projective connection ω' *(in the Euclidean space $\mathbb{R}^{n+1}$) if it is* inhomogeneous*,* horizontal *and* right distinguished equivariant*, i.e., such that for all projective frame field $\mathfrak{F} \in \Gamma_{n+1}(\mathfrak{F}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}))$, for all* horizontal *vector field $\underline{\mathfrak{v}} \in \chi(\mathbb{R}^{n+1})$ and for all projective distinguished diffeomorphism $f \in PDiff^r_{loc.}(\mathbb{R}^{n+1}, \mathcal{F}, P_0)$ preserving ξ, i.e., $f_*(\xi) = a\,\xi$ where a is a smooth function on M, we have*

1. *for all $\mathfrak{v} \in \chi(M)$ we have $i_{\mathfrak{v}}\omega(\mathfrak{F}) \in \Gamma_{n+1}(End(n+1,\mathbb{R})/\{\mathbb{R}\,\mathbb{1}\})$, i.e., $i_{\mathfrak{v}}\omega(\mathfrak{F})$ has values in the Lie algebra of $PGL(n+1,\mathbb{R})$,*

2. *for all $\mathfrak{v} \in \chi(M)$ and for all $\mathfrak{F} \in \Gamma_{n+1}(Aff(n,\mathbb{R})) \subset \Gamma_{n+1}(\mathfrak{F}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1}))$ we have $i_{\mathfrak{v}}\omega(\mathfrak{F}) \in \Gamma_{n+1}(\mathbb{R}^n \rtimes gl(n,\mathbb{R}))$, i.e., $i_{\mathfrak{v}}\omega(\mathfrak{F})$ has values in the Lie algebra of $Aff(n,\mathbb{R})$,*

3. *$i_{\underline{\mathfrak{v}}} R^*_f(\omega)(\mathfrak{F}) = Ad(Tf^{-1})\, i_{\underline{\mathfrak{v}}}\omega(\mathfrak{F})$, i.e., ω is* 'right distinguished equivariant' *with respect to f, and*

4. *$i_{\underline{\mathfrak{v}}}\omega = 0$ only if $\underline{\mathfrak{v}} = 0$.*

Moreover, for any Euclidean basis fields $\mathfrak{B} \in \mathfrak{F}$ and $\mathfrak{B}' \in \mathfrak{F}'$, the right action of $\mathfrak{F}$ on ω satisfies the relations

$$R^*_{\mathfrak{F}}(\omega) = Ad(\mathfrak{B}^{-1})\,\omega\,. \tag{3.22}$$

Moreover, we can make a few comments on this definition.

1. We have no notion such as the notions of horizontal or inhomogeneous connection in the Ehresmann's definition. Actually, the soldering between F and B removes somehow the notion of horizontality since, within this context, horizontality with respect to B would

mean "horizontality" with respect to F as well. Moreover, the inhomogeneity is also void of meaning since the solderings between B and F are homeomorphisms differing from the submersions such as W.

Nevertheless, it seems to remain a difference between the right equivariances. In the Ehresmann's definition, the right equivariance is with respect to G identified with $PGL(n+1,\mathbb{R})$ whereas the right distinguished equivariance given in Definition 7 is with respect to a particular pseudogroup of diffeomorphisms f which are distinguished relative to the foliation $\mathcal{F}$. Actually, each diffeomorphism in this pseudogroup defines a bundle map such as f_* (or f^*) in the (co-)tangent bundle connecting the Lie group $Stab([\xi])$ over a point $p \in M$ to another stabilizer $Stab([\xi'])$ at $p' \neq p$. Only $PGL(n{+}1,\mathbb{R})$ acts transitively on this set of stabilizers and therefore the bundle maps defined by the distinguished diffeomorphisms expressed the left action of $PGL(n+1,\mathbb{R})$ on the stabilizers. Hence, in fact, we have equivalent right equivariances and no ground, fundamental differences.

2. In the condition *3* of Definition 7, $\mathfrak{F}$ is considered as a field with values in a Lie group, namely, $\mathit{Aff}(n,\mathbb{R})$. This expresses the dual aspect of a projective frame when we consider that a frame acts on the left on another one as in formula (3.21). Then, this condition *3* is the equivalent of the condition *1* in the Ehresmann's definition (p. 23). Also, we consider that the projective frames $\mathfrak{F} = \{[\mathfrak{v}_0], \ldots, [\mathfrak{v}_{n+1}]\}$ are elements of $\Gamma_{n+1}(\mathit{Aff}(n,\mathbb{R}))$ whenever $[\mathfrak{v}_0] = [\xi]$ and $\mathfrak{v}_i(p) \in \mathfrak{P}_1(p) \simeq T_p\mathcal{J}_0$. Besides, let $\mathfrak{B}$ be an element in the class $\mathfrak{F}$. This basis $\mathfrak{B}$ can be represented, for instance, by a $(n+1)$-square matrix $\mathfrak{M}(p)$ at $p \in M$ such that

$$\mathfrak{M}(p) \equiv \left(\begin{array}{c|c} 1 & 0 \;\ldots\ldots\; 0 \\ \hline r(p) & M_{n\times n}(p) \end{array}\right) \in SL(n+1,\mathbb{R})\,.$$

where $r(p)$ is a vector in $\mathbb{R}^n$, $M_{n\times n}(p)$ is a n-square matrix in $SL(n,\mathbb{R})$. Then, we consider that $\mathfrak{M}(p')$ at $p' \neq p$ is represented by a matrix such that $\mathfrak{M}(p') = P\,\mathfrak{M}(p)P^{-1}$ where P is a change of basis matrix. Now, we consider a curve $c(t)$ in M such that $c(0) = p$, $p' = c(t)$ and where $P \in SL(n+1,\mathbb{R})$ is depending on $t \in [0,1]$ for instance. We deduce easily that $\mathfrak{M}(c(t))$ satisfied the following system of ODEs

$$\dot{\mathfrak{M}} = \mathfrak{U}\,\mathfrak{M} - \mathfrak{M}\,\mathfrak{U} \equiv ad_{\mathfrak{U}}(\mathfrak{M})\,, \tag{3.23}$$

where $\mathfrak{U} \equiv \dot{P}P^{-1} \in sl(n+1,\mathbb{R})$. Then, let $\mathfrak{X}$ be another $(n+1)$-square matrix depending on t such that

$$\dot{\mathfrak{X}} = \mathfrak{U}\,\mathfrak{X}\,. \tag{3.24}$$

As a result, if $\mathfrak{X}_1$ and $\mathfrak{X}_2$ are two linearly independent solutions of (3.24) with nonvanishing determinants, we obtain that $\mathfrak{M} \equiv \mathfrak{X}_1\,\mathfrak{X}_2^{-1}$ is a solution of (3.23) (we have a so-called *non-linear superposition principle*. See Winternitz and *al.*; for instance [Win82]). Remarquably, we can take also two linearly independent invertible solutions $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of (3.23) to obtain a third solution $\mathfrak{M}_3 \equiv \mathfrak{M}_1\,\mathfrak{M}_2^{-1}$. But also, the matrices $\mathfrak{M}$ can represent frames, $ad_{\mathfrak{U}}$ the projective connection 1-form contracted by $\mathfrak{v} \equiv \dot{p} \in \chi(M)$ and $\dot{\mathfrak{M}}$ the covariant derivative with respect to $\mathfrak{v}$. The matrices $\mathfrak{M}$ can be interpreted as giving the inhomogeneous coordinates of a point in the Grassmann manifold $Gr_{n+1}(\mathbb{R}^{2(n+1)})$ of $(n+1)$-dimensional planes in $\mathbb{R}^{2(n+1)}$ and $\mathfrak{X}_1$ and $\mathfrak{X}_2$ as giving its homogeneous coordinates. From this remark, this condition means that $ad_{\mathfrak{U}} \equiv i_{\mathfrak{v}}\omega$ at any given $p \in M$ defines a (group representation depending) *Maurer-Cartan form* of the Lie group $SL(n+1,\mathbb{R})/Z \simeq PSL(n+1,\mathbb{R})$ where $Z \equiv \{\pm 1\}$ if n is odd and $Z \equiv \{1\}$ otherwise.

3. The projective connection ω is defined on $\mathfrak{F}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$ and not on $\mathfrak{B}_{GL(n+1,\mathbb{R})}(T\mathbb{R}^{n+1})$. This is necessary because the projective map $\wp$ covered by the projective distinguished diffeomorphism f is insensitive to scale changes on Euclidean bases $\mathfrak{B}$. In addition, it matters to notice that this scale invariance differs from the one due to the representations $\omega_{\mathfrak{B}}$ of ω. It is due to the definition of $PGL(n+1,\mathbb{R})$ as the coset of $GL(n+1,\mathbb{R})$ by $\mathbb{R}^*$.

4. The condition *3* in Definition 7 must be clearly interpreted in terms of the components of the homogeneous $(n+1)$-square matrix $\mathcal{G}$ defined by $\omega(\mathfrak{F})$. Then, firstly, we have $\omega(\mathfrak{F}) \equiv \sum_{\alpha,\beta=0}^{n} \mathcal{G}_\beta^\alpha\,\mathfrak{i}_\alpha \otimes \mathfrak{i}^{*,\beta}$ where $\mathcal{G}_\beta^\alpha \equiv \mathfrak{v}^{*,\alpha}(\mathcal{G}\,.\,\mathfrak{v}_\beta)$ and $\mathfrak{F} \equiv [\mathfrak{B}]$. And, secondly, we denote

by $\mathfrak{B}' \equiv f_*(\mathfrak{B}) \equiv \{\mathfrak{v}'_0, \mathfrak{v}'_1, \ldots, \mathfrak{v}'_n\}$ the transformed representing basis ascribed to $\mathfrak{F}' \equiv [\mathfrak{B}']$. Then, we have, in particular, $\omega(\mathfrak{F}') \equiv \sum_{\alpha,\beta=0}^{n} \mathcal{G}'^{\alpha}_{\beta}\, \mathfrak{i}_\alpha \otimes \mathfrak{i}^{*,\beta}$ where $\mathcal{G}'^{\alpha}_{\beta} \equiv \mathfrak{v}'^{*,\alpha}(\mathcal{G} \,.\, \mathfrak{v}'_\beta)$. And, moreover, if we assume also that $\mathfrak{v}_0 = \xi$ and $\mathfrak{v}'_0 = \xi' \equiv f_*(\xi) = Tf(\xi) \circ f^{-1} = a\,\xi$ ($\Longrightarrow Tf^i_0 = 0$), then, the relation in the condition *3* of Definition 7 means that the relations

$$i_{T_pf(\underline{\mathfrak{v}})}\mathcal{G}'^{i}_{j}(f(p)) = \sum_{r,s=1}^{n} T_pf^s_j\,(i_{\underline{\mathfrak{v}}}\mathcal{G}^r_s(p))\,T_{f(p)}f^{-1,i}_r \tag{3.25a}$$

and

$$i_{T_pf(\underline{\mathfrak{v}})}\mathcal{G}'^{0}_{0}(f(p)) = i_{\underline{\mathfrak{v}}}\mathcal{G}^0_0(p) \tag{3.25b}$$

hold in the canonical Euclidean basis $\{\mathfrak{i}_0, \ldots, \mathfrak{i}_n\}$ and cobasis $\{\mathfrak{i}^{*,0}, \ldots, \mathfrak{i}^{*,n}\}$, where $i, j = 1, \ldots, n$. Indeed, Tf is completely reducible (but not reduced) and block triangular in this basis ($\pi = \mathfrak{v}^{*,0}$, $f^*(\pi) = a\,\pi$ and $\mathfrak{v}'^{*,0} = \pi' = a^{-1}\,\pi$). In particular, if for all $\alpha = 0, \ldots, n$ we have $f_*(\mathfrak{v}_\alpha) \equiv \mathfrak{v}'_\alpha = \mathfrak{v}_\alpha \Longleftrightarrow \mathfrak{v}_\alpha(f(p)) = T_pf(\mathfrak{v}_\alpha(p)) = \sum_{\beta=0}^{n} T_pf^\beta_\alpha\,\mathfrak{v}_\beta(p)$ ($\not\Longrightarrow T_pf = \mathbb{1}$), then, we obtain (the prime mark disappears):

$$i_{T_pf(\underline{\mathfrak{v}})}\mathcal{G}^{i}_{j}(f(p)) = \sum_{r,s=1}^{n} T_pf^s_j\,(i_{\underline{\mathfrak{v}}}\mathcal{G}^r_s(p))\,T_{f(p)}f^{-1,i}_r\,, \tag{3.26a}$$

$$i_{T_pf(\underline{\mathfrak{v}})}\mathcal{G}^{0}_{0}(f(p)) = i_{\underline{\mathfrak{v}}}\mathcal{G}^0_0(p)\,. \tag{3.26b}$$

Also, if $\{\mathfrak{g}_a,\, a = 1, \ldots, K\}$ are the generators of the Lie algebra of $PGL(n+1, \mathbb{R})$ such that $\mathfrak{g}_a \equiv \sum_{\alpha,\beta=0}^{n} \mathfrak{g}^{\alpha}_{a,\beta}\, \mathfrak{i}_\alpha \otimes \mathfrak{i}^{*,\beta}$, then, $\omega \equiv \sum_{a=1}^{K} \omega^a\, \mathfrak{g}_a$, and then, we deduce also that

$$\sum_{a=1}^{K} R^*_f(\omega)^a \mathfrak{g}_a = \sum_{b=1}^{K} Ad(Tf^{-1})(\mathfrak{g}_b)\,\omega^b = \sum_{b,c=1}^{K} \omega^b\, Ad(Tf^{-1})^c_b\,\mathfrak{g}_c\,. \tag{3.27}$$

Remark 5. *At this point, we ought to note that there is a conformal aspect ($f^*(\pi) = a\,\pi$ and $f_*(\xi) = a\,\xi$) in this definition coming from the notion of projective distinguished maps and this has certainly been an important source of confusion in the historical developments, understanding or diffusion of the differential projective geometry formalism. Indeed, O. Veblen, B. Hoffmann, D. van Dantzig, T.Y. Thomas, J.A. Schouten, J. Haantjes, introduced this projective geometry* *starting explicitly from such conformal equivariance contrary to É. Cartan. And we could be enforced to think that projective geometry is a conformal geometry.*

Clearly, the projective geometry is not a conformal geometry even if in these two geometries scaling transformations appear explicitly.[28] *Indeed, the projective geometry is about linking transformations (connections) from point to point of wire-spoked congruences of vector lines centered and attached to given points whereas the conformal geometry is about linking transformations, from point to point, of congruences of nested spheres centered and attached to given points (see [Car22b, Car24a]). In these two situations, the fact that we have congruences involves to consider conformal transformations to pass from a vector line to another vector line of the same congruence attached to a given point, or homothetic transformations to pass from a sphere to another one both elements of the same congruence of nested spheres centered at a given point.*

Then, the scaling aspect is associated with the definition of the congruences in the projective geometry and not to the linking transformations between these congruences whereas, on the contrary, it is associated with the linking transformations and not to the definition of the congruences in the conformal geometry.

*Additionally, these two geometries are somehow duals to one another with a duality defined by '*reciprocation*' (or, more precisely, '*polar reciprocation*') and which transforms radii to spheres or reciprocally [Brü00, Cre05] [CG67, §6.1 "Reciprocation", pp.132–136] [Ogi69, pp.107–110] [Wen83, pp.1–6].*

Then, we define the following.

Definition 8. *We denote by* $PDiff^r_{loc.}(M, \mathcal{F}, \mathcal{J}_0, \xi)$ *the subset of local diffeomorphisms* $f \in PDiff^r_{loc.}(M, \mathcal{F}, \mathcal{J}_0)$ *such that there exists a* C^r *function* a *on any open* $\mathcal{U}$ *such that*

$$f_*(\xi) = a\,\xi\,. \tag{3.28}$$

And lastly, we can generalize the precedent definition for a projective connection.

[28] We can note that it is considered erroneously as a conformal geometry because we have a "conformal factor" in most transformations; a problem certainly due to an inappropriate terminology used for what we should call the 'scaling factors.'

Definition 9. *A class C^r map $\omega : \Gamma_{n+1}(\mathfrak{F}_{GL(n+1,\mathbb{R})}(TM)) \longrightarrow \Gamma_{n+1}(End_{GL(n+1,\mathbb{R})}(TM) \otimes T^*M)$ is a* 'projective connection ω' *defined on the C^r manifold M modeled on $\mathbb{R}P^n$ if it is* inhomogeneous*,* horizontal *and* right distinguished equivariant*, i.e., such that for all projective frame field $\mathfrak{F} \in \Gamma_{n+1}(\mathfrak{F}_{GL(n+1,\mathbb{R})}(TM))$, for all* horizontal *vector field $\underline{\mathfrak{v}} \in \chi(M)$ and for all projective distinguished diffeomorphism $f \in PDiff^r_{loc.}(M, \mathcal{F}, \mathcal{J}_0, \xi)$ we have*

1. *for all $\mathfrak{v} \in \chi(M)$ we have $i_{\mathfrak{v}}\omega(\mathfrak{F}) \in \Gamma_{n+1}(End(n+1,\mathbb{R})/\{\mathbb{R}\,\mathbb{1}\})$, i.e., $i_{\mathfrak{v}}\omega(\mathfrak{F})$ has values in the Lie algebra of $PGL(n+1,\mathbb{R})$,*

2. *for all $\mathfrak{v} \in \chi(M)$ and for all $\mathfrak{F} \in \Gamma_{n+1}(Aff(n,\mathbb{R})) \subset \Gamma_{n+1}(\mathfrak{F}_{GL(n+1,\mathbb{R})}(TM))$ we have $i_{\mathfrak{v}}\omega(\mathfrak{F}) \in \Gamma_{n+1}(\mathbb{R}^n \rtimes gl(n,\mathbb{R}))$, i.e., $i_{\mathfrak{v}}\omega(\mathfrak{F})$ has values in the Lie algebra of $Aff(n,\mathbb{R})$,*

3. *$i_{\underline{\mathfrak{v}}} R^*_f(\omega)(\mathfrak{F}) = Ad(Tf^{-1})\, i_{\underline{\mathfrak{v}}}\omega(\mathfrak{F})$, i.e., ω is* 'right distinguished equivariant' *with respect to f, and*

4. *$i_{\underline{\mathfrak{v}}}\omega = 0$ only if $\underline{\mathfrak{v}} = 0$.*

Moreover, for any Euclidean basis fields $\mathfrak{B} \in \mathfrak{F}$ and $\mathfrak{B}' \in \mathfrak{F}'$, the right action of $\mathfrak{F}$ on ω satisfies the relations

$$R^*_{\mathfrak{F}}(\omega) = Ad(\mathfrak{B}^{-1})\,\omega\,. \tag{3.29}$$

2. *From Euclidean connections to projective connections and their associated covariant derivatives*

From now and throughout, we identify P_0 and $\mathcal{J}_0$ via a system of local charts on $\mathcal{J}_0$ endowed with the topology induced by the topology on M. Thus, we have $P_0 \simeq \mathcal{J}_0$ and locally $M \simeq_{loc.} \mathbb{R}^{n+1}$. Then, we denote by $\not\nabla$ a covariant derivative associated with a Euclidean connection $\not\omega$; the latter defined *over* $\mathbb{R}P^n$ (*i.e.*, locally over P_0), and thus, *inhomogeneous*. Then, we have in the basis field $\mathfrak{B} = \{\mathfrak{v}_0, \ldots, \mathfrak{v}_n\}$:

$$\not\nabla_{\mathfrak{u}}\mathfrak{v}_\alpha = (i_{\mathfrak{u}}\not\omega^\beta_\alpha)\mathfrak{v}_\beta\,, \tag{3.30}$$

where $\mathfrak{u}$ is any vector field in $\chi(M)$. Besides, we can put in correspondence the notations and the notions defined by Ehresmann and those used in the present context with, for instance, the three-dimensional leaves $\mathcal{J} \subset M$ of the four dimensional manifold M modeled on $\mathbb{R}P^3$:

B	$\longleftrightarrow$	$\mathcal{J}_0$
F	$\longleftrightarrow$	$\mathbb{R}P^3$
o (origin of F)	$\longleftrightarrow$	$[\xi_0] \equiv (p_0, [1,0,0])$ (origin of $\mathbb{R}P^3$)
G	$\longleftrightarrow$	$PGL(4,\mathbb{R})$
G'	$\longleftrightarrow$	$\mathit{Aff}(3,\mathbb{R}) \simeq \mathit{Stab}([\xi])$
$h_z(\in H_z) \simeq f^N(\in F^N/\triangle^N)$	$\longleftrightarrow$	Projective transformations identified with $L_{\mathfrak{B}}$ with $\mathfrak{B} \in \mathfrak{F}$ and $\mathfrak{F}$ a projective frame with $N = n+2 = 5$ vector lines in $\mathbb{R}^4$. Moreover, we know that a pair (see the "vector lines" f^N in footnote 17) of five distinct vector lines in $\mathbb{R}^4$ define, nevertheless up to a factor, any element of $GL(4,\mathbb{R})$, and thus, define completely an element of $PGL(4,\mathbb{R})$ (see [Die78, pp. 134–135, 'N-transitivity']).
$h'_z \in H'_z$	$\longleftrightarrow$	Conformal transformations identified with the right action $R_{\mathfrak{B}}$ where $\mathfrak{F} \ni \mathfrak{B} = \{\mathfrak{v}_0, \ldots, \mathfrak{v}_n\} \in \mathit{Aff}(n,\mathbb{R})$ with $\mathfrak{v}_0 \equiv \xi \notin \mathfrak{P}_1$ and $\mathfrak{v}_i \in \mathfrak{P}_1$

But, there are different ways to embed $\mathit{Aff}(n,\mathbb{R})$ into $GL(n+1,\mathbb{R})$ depending on how the vector spaces $T_p\mathcal{J}_0 \simeq \mathfrak{P}_1(p)$ are parallel transported along curves in $\mathcal{J}_0$. As we noticed, the structural group of $T\mathbb{R}P^n$ is the sub-group $\mathit{Aff}(n,\mathbb{R}) \subset PGL(n+1,\mathbb{R})$ corresponding to the "homogeneous" Lie group $\mathbb{R}^*\times \mathit{Aff}(n,\mathbb{R}) \subset GL(n+1,\mathbb{R})$ acting on $T\mathbb{R}^{n+1}$. Thus, if $\not\omega$ is also a projective connection associated with $\not\nabla$, then $\not\omega_{\mathfrak{B}}$ with $\mathfrak{B} \in \mathit{Aff}(n,\mathbb{R})$ must be a $\rho(\mathbb{R}\oplus \mathit{aff}(n,\mathbb{R}))$-valued *horizontal* 1-form over $\mathbb{R}P^n$ defined *modulo* any $\rho(\mathbb{R})$-valued *horizontal* 1-form over $\mathbb{R}P^n$, where ρ is a faithful representation of $\mathbb{R}\oplus \mathit{aff}(n,\mathbb{R})$ in the associative $\mathbb{R}$-algebra M_{n+1} of $(n+1)$-

square matrices, *i.e.*, $\rho : \mathbb{R}\oplus \mathit{aff}(n,\mathbb{R}) \longrightarrow M_{n+1} \simeq gl(n+1,\mathbb{R})$ is a $\mathbb{R}$-monomorphism.

Moreover, we must determine the representation ρ such that, if $r \in \mathbb{R}$, we have $\rho(r) \equiv r\,\mathbb{1}$, where $\mathbb{1}$ is the identity morphism on $T\mathbb{R}^{n+1} \simeq_{loc.} TM$. Moreover, it indicates that the projective connection ω is such that $\omega_{\mathfrak{B}} \equiv (\omega_\alpha^\beta)$ in $\mathbb{R}^{n+1} \simeq_{loc.} M$ is associated univocally with an *affine connection* in $\mathbb{R}^n$, *i.e.*, a pair (ω_k^0, ω_i^j) where (ω_i^j) is an Euclidean connection in $\mathbb{R}^n$ and (ω_k^0) is a vector-valued 1-form in $\mathbb{R}^n$ $(i,j,k = 1,\ldots,n)$.

Then, denoting any set of germs of smooth local maps in $C^\infty(\mathbb{R}P^n, X)$ from $\mathbb{R}P^n$ to any smooth manifold X by $\Gamma_n(X)$ and denoting by $\mathcal{O}_{\mathbb{R}P}$ the presheaf of rings of germs of the real smooth functions defined on $\mathbb{R}P^n$, *i.e.*, of the local inhomogeneous real smooth functions, we define the following.

Definition 10. *The pairs of covariant projective derivatives and Yano-Ishihara (projecting) 1-forms (∇,π) and (∇',π') are* 'equivalent,' *i.e., $(\nabla,\pi) \sim (\nabla',\pi')$, if and only if there exists a local differentiable section $\mathfrak{C}$ of the principal frame bundle $\mathcal{F}_{GL(n+1,\mathbb{R})}(\mathbb{R}P^n, TM)$ of bases fields $\mathfrak{B}$ of TM over $\mathbb{R}P^n$ (or, locally, $\mathcal{J}_0$), i.e., $\mathfrak{C} \in \Gamma_n(GL(n+1,\mathbb{R}))$, such that for all smooth basis field $\mathfrak{B}$ we have*

1. $\nabla'_{\mathfrak{u}}\mathfrak{B} = R_{\mathfrak{C}^{-1}}\nabla_{\mathfrak{u}}(R_{\mathfrak{C}}\mathfrak{B})\,,$ (3.31a)
2. $\pi'(\mathfrak{B}) = \pi(R_{\mathfrak{C}}\mathfrak{B})\,,$ (3.31b)

where $\mathfrak{u}$ is any vector field in $\Gamma_n(TM) \equiv \chi(\mathbb{R}P^n)$ and $\pi(\mathfrak{B}) \equiv \{\pi(\mathfrak{v}_0),\ldots,\pi(\mathfrak{v}_n)\}$, $\nabla\mathfrak{B} \equiv \{\nabla\mathfrak{v}_0,\ldots,\nabla\mathfrak{v}_n\}$ and $R_{\mathfrak{C}}\mathfrak{B} \equiv \{\mathfrak{C}\mathfrak{v}_0,\ldots,\mathfrak{C}\mathfrak{v}_n\}$.

It is extremely important to note that $\mathfrak{C}$ is a change of basis matrix (field), and thus, $\mathfrak{C}$ does not refer to representations in specific bases of TM or also to local diffeomorphisms of M. In other words, $\mathfrak{C}$ can only be applied to bases and not to individual vectors. In other words, we consider that bases act on bases and not on a single vector, all the more so as we need beforehand to know the action of a given basis on another basis to deduce the action on a single vector. Hence, considering an image vector $\mathfrak{C}(\mathfrak{v})$ would be just an abuse of definition or notation whereas $\mathfrak{C}(\mathfrak{B})$ is not for any basis $\mathfrak{B}$ in TM. Roughly speaking, the two relations in the definition above involves the whole of the following $n+1$ relations

$$\nabla'_{\mathfrak{u}}\mathfrak{v}_\alpha = \mathfrak{C}^{-1}(\nabla_{\mathfrak{u}}(\mathfrak{C}(\mathfrak{v}_\alpha))) \equiv Ad(\mathfrak{C}^{-1})(\nabla_{\mathfrak{u}})\mathfrak{v}_\alpha, \tag{3.32a}$$

$$\pi'(\mathfrak{v}_\alpha) = \pi(\mathfrak{C}(\mathfrak{v}_\alpha)), \tag{3.32b}$$

In more detail, the condition *1* in the precedent definition means the following. If $\mathfrak{C}^{-1}(\nabla_{\mathfrak{u}}\mathfrak{C}(\mathfrak{v}_\alpha))$ $= \nabla'_{\mathfrak{u}}\mathfrak{v}_\alpha$ then, from the definition $\mathfrak{C}(\mathfrak{v}_\alpha) = \mathfrak{C}^\beta_\alpha\mathfrak{v}_\beta = \mathfrak{v}'_\alpha$, we have $\mathfrak{C}^{-1}(\nabla_{\mathfrak{u}}(\mathfrak{C}^\beta_\alpha\mathfrak{v}_\beta)) = \nabla'_{\mathfrak{u}}\mathfrak{v}_\alpha$ $\Longleftrightarrow \mathfrak{C}^{-1}(\mathfrak{v}_\beta)(i_{\mathfrak{u}}d\mathfrak{C}^\beta_\alpha) + \mathfrak{C}^\beta_\alpha\mathfrak{C}^{-1}(\nabla_{\mathfrak{u}}\mathfrak{v}_\beta) = \nabla'_{\mathfrak{u}}\mathfrak{v}_\alpha$. But, with $\nabla_{\mathfrak{u}}\mathfrak{v}_\beta = \mathfrak{v}_\mu\,\omega^\mu_\beta(\mathfrak{u})$, we deduce that $\mathfrak{C}^{-1}(\mathfrak{v}_\beta)(i_{\mathfrak{u}}d\mathfrak{C}^\beta_\alpha) + \mathfrak{C}^\beta_\alpha\mathfrak{C}^{-1}(\nabla_{\mathfrak{u}}\mathfrak{v}_\beta) = \nabla'_{\mathfrak{u}}\mathfrak{v}_\alpha \Longleftrightarrow \mathfrak{C}^{-1}(\mathfrak{v}_\gamma)\,i_{\mathfrak{u}}(d\mathfrak{C}^\gamma_\alpha + \mathfrak{C}^\beta_\alpha\,\omega^\gamma_\beta) = \nabla'_{\mathfrak{u}}\mathfrak{v}_\alpha$. We conclude that ${\omega'}^\beta_\alpha \equiv (\mathfrak{C}^{-1})^\beta_\gamma\,d\mathfrak{C}^\gamma_\alpha + (\mathfrak{C}^{-1})^\beta_\gamma\,\omega^\gamma_\mu\,\mathfrak{C}^\mu_\alpha$ in the unique common basis $\mathfrak{B} \equiv \{\mathfrak{v}_1,\dots,\mathfrak{v}_n\}$ and its dual. Or, equivalently, we have

$$\omega'_{\mathfrak{B}} = \mathfrak{C}^{-1}\,\omega_{\mathfrak{B}}\,\mathfrak{C} + \mathfrak{C}^{-1}.\,d\mathfrak{C} \tag{3.33}$$

with (3.32b) $\Longleftrightarrow (\omega',\pi') \sim (\omega,\pi)$.

Remark 6. *This result differs strongly from the one we obtain with a gauge transformation in physics thought the formula is very similar. Indeed, in a gauge transformation we have only one covariant derivative at hand,* viz., ∇. *The computation is well-known but it is related, in fact, to a change of coordinate chart. Indeed, given two local trivialization charts* (U_1,ϕ_1) *and* (U_2,ϕ_2) *of a tangent bundle* $\flat : TM \longrightarrow M$ *where* $\dim M = n$*, the* U_i*'s (*$i=1,2$*) are opens in* M *and* $\phi_i : \flat^{-1}(U_i) \longrightarrow U_i \times \mathbb{R}^n$ *are homeomorphisms, then, to a vector* $\mathfrak{v} \in \flat^{-1}(U_i)$ *it corresponds a vector* $v_i \in \mathbb{R}^n$ *such that* $v_i = pr_2 \circ \phi_i(\mathfrak{v}) \equiv \rho_i(\mathfrak{v})$*. Then, assuming that* $U_1 \cap U_2 \neq \varnothing$ *and* $\mathfrak{v}$ *is a smooth vector field on* $U_1 \cap U_2$*, i.e.,* $\mathfrak{v} \in \chi(U_1 \cap U_2)$*, then, we have* $v_1 = t_{1,2}(v_2)$ *where* $t_{1,2} : U_1 \cap U_2 \longrightarrow GL(n,\mathbb{R})$ *is a transition function assumed to be smooth and which is the Jacobian matrix of the change of coordinates. Moreover, the covariant derivative* ∇ *is defined on* $\mathbb{R}^n$ *and applies on vector fields such as the* v_i*'s. Similarly, we have* $\mathcal{B}_i \equiv \rho_i(\mathfrak{B}) \equiv \{\rho_i(\mathfrak{v}_1),\dots,\rho_i(\mathfrak{v}_n)\}$ *and two corresponding Levi-Civita connections* $\gamma_{i,\mathfrak{B}} \equiv \gamma_{\mathcal{B}_i}$ *corresponding to a unique Levi-Civita connection* ω *on* M *evaluated at the unique basis* $\mathfrak{B}$*.*

Then, we have $\nabla v_{i,\alpha} = (\gamma_{i,\mathfrak{B}})^\beta_\alpha\, v_{i,\beta}$*, and therefore, we have*

$$\begin{aligned}(\gamma_{2,\mathfrak{B}})^\beta_\alpha\, v_{2,\beta} &= \nabla v_{2,\alpha}\\ &= \nabla((t_{2,1})^\beta_\alpha v_{1,\beta})\\ &= d(t_{2,1})^\beta_\alpha v_{1,\beta} + (t_{2,1})^\beta_\alpha \nabla v_{1,\beta}\\ &= d(t_{2,1})^\beta_\alpha v_{1,\beta} + (t_{2,1})^\mu_\alpha (\gamma_{1,\mathfrak{B}})^\beta_\mu\, v_{1,\beta}\,.\end{aligned}$$

Thus, we obtain $(\gamma_{2,\mathfrak{B}})^\mu_\alpha = d(t_{2,1})^\beta_\alpha (t^{-1}_{2,1})^\mu_\beta + (t_{2,1})^\nu_\alpha (\gamma_{1,\mathfrak{B}})^\beta_\nu\, (t^{-1}_{2,1})^\mu_\beta$ *or, equivalently,*

$$\gamma_{2,\mathfrak{B}} = t^{-1}_{2,1}\,\gamma_{1,\mathfrak{B}}\, t_{2,1} + t^{-1}_{2,1}\, d(t_{2,1}) \quad \Longleftrightarrow \quad \gamma_{\mathcal{B}_2} = t^{-1}_{2,1}\,\gamma_{\mathcal{B}_1}\, t_{2,1} + t^{-1}_{2,1}\, d(t_{2,1})\,. \tag{3.34}$$

Hence, in this case, we change the basis since we pass from $\gamma_{\mathcal{B}_1}$ *to* $\gamma_{\mathcal{B}_2}$ *but not the covariant derivative and the connection contrary to the formula* (3.33)*. Also, the formula above should be clearly in conflict with a formula "*$R^*_{t_{2,1}}(\gamma_{\mathcal{B}_1}) = Ad(t^{-1}_{2,1})\,\gamma_{\mathcal{B}_1}$*" analogous to the formula* (3.29)*, i.e.,* $R^*_{\mathfrak{F}}(\omega) = Ad(\mathfrak{C}^{-1})\,\omega$*, contrary to* (3.33) *since* $R^*_{t_{2,1}}(\gamma_{\mathcal{B}_1}) = \gamma_{\mathcal{B}_2} \neq Ad(t^{-1}_{2,1})\,\gamma_{\mathcal{B}_1}$ *whereas* $R^*_{\mathfrak{F}}(\omega)_{\mathfrak{B}} = Ad(\mathfrak{C}^{-1})\,\omega_{\mathfrak{B}} \equiv \omega_{\mathfrak{B}'} \neq \omega'_{\mathfrak{B}}$ *where* $\mathfrak{B}' = R_{\mathfrak{C}}(\mathfrak{B})$*.*

Remark 7. *The previous example shows a practical difficulty which is often hidden or absent in the mathematical literature although or because linked to an extremely, elementary and basic aspect, viz, the representation of a tensor by coordinates in a given basis. This is manifested explicitly in an almost systematic problem occurring between the physics literature and the mathematical literature about the subscript notation of tensors with its plethora of indices, accompanied by its batch of ambiguities, troubleshootings and contradictions about the intrinsic status or not of the mathematical objects manipulated along some effective computations. If it is absolutely, mathematically well-justified to avoid to write formulas with subscript/superscript notation and numerical indices in geometry, it remains nevertheless a problem with certain formulas involving connection forms. Indeed, the connection forms are defined on principal frame bundles, and thus, their "values" depend explicitly on given bases.*

Then, we can be faced typically to two situations: if the formulas we consider define "intrinsic" tensors then indices can be avoid, but, on the contrary, a basis must be specified and the problem of numerical indices appears and cannot be avoided.

Actually, numerical indices can always be avoided, but not a particular set of non-numerical indices, namely, the symbol or the letter ascribed to the naming of a basis. More precisely, given a basis $\mathfrak{B} = \{\mathfrak{v}_1, \ldots, \mathfrak{v}_n\}$*, its dual cobasis* $\mathfrak{B}^* = \{\mathfrak{v}^{*1}, \ldots, \mathfrak{v}^{*n}\}$ *and a vector field* $\mathfrak{u}$*, then, we can define a map* $\mathcal{B} : \mathfrak{u} \in TM \longrightarrow \mathfrak{u}_{\mathfrak{B}} \in \chi(\mathbb{R}^n)$ *where* $\mathfrak{u}_{\mathfrak{B}}$ *is a one column matrix with components* $\mathfrak{u}^i_{\mathfrak{B}} \equiv \mathfrak{v}^{*i}(\mathfrak{u})$ *with numerical indices* $i = 1, \ldots, n$*. Hence, we have, at least, the subscript notation* $\mathfrak{u}_{\mathfrak{B}}$*. In addition, if we change the basis for the representation of the vectors, i.e., if we pass from* $\mathfrak{B}$ *to* $\mathfrak{B}' = R_{\mathfrak{C}}(\mathfrak{B})$*, then, we have* $\mathfrak{u}_{\mathfrak{B}'} = K \,.\, \mathfrak{u}_{\mathfrak{B}}$ *where* K *defined from* $\mathfrak{C}$ *is the change of basis matrix which must be jointly, absolutely specified.*

*Then, if we consider tensor formulas defined from a given connection form and if we juggle with, for instance, both Euclidean geometry and projective geometry, tensors which are "intrinsic" to the Euclidean geometry are not necessarily intrinsic within the context of projective geometry, and then, bases must be specified to express these projective tensors in the given Euclidean geometry. This is not just a scholar problem. Indeed, for instance, they had strong, conceptual and historical repercussions in the emergence of the notion of '*projective analytic tensors*' introduced historically by É. Cartan in his seminal 1935 Moscow paper [Car35, unfortunately only in french]. These categories of tensors were entirely absent from the works on the projective geometry of O. Veblen, B. Hoffmann, J.M. Thomas, J.A. Schouten, D. van Dantzig and J. Haantjes who considered only Euclidean (covariant or contravariant) tensors but noticed for the first time by É. Cartan. We can notice also historically that É. Cartan and O. Veblen could not help but give themselves little kicking claws as it can be shown reading, for instance, the ironical footnotes of Cartan in his 1935 Moscow paper. Hence, the general situation is an intermediate one: non-numerical subscript indices cannot be avoided in particular "intrinsic" situations.*

Remark 8. *The formula* (3.31a) *can be expanded and we obtain the other following expressions for the basis vectors* $\mathfrak{v}_\alpha \in \mathfrak{B}$*:*

$$\nabla'_{\mathfrak{u}}\mathfrak{v}_\alpha = \nabla_{\mathfrak{u}}\mathfrak{v}_\alpha + (\mathfrak{C}^{-1} \,.\, i_{\mathfrak{u}} d\mathfrak{C})\mathfrak{v}_\alpha \,, \tag{3.35}$$

where $\mathfrak{C}^{-1} \,.\, i_{\mathfrak{u}} d\mathfrak{C} \in \Gamma_n(gl(n+1, \mathbb{R}))$*.*

Remark 9. *From this definition, we note that any given* inhomogeneous *connection* ω *is defined* *as a field over the principal bundle $\mathcal{F}_{GL(n+1,\mathbb{R})}(\mathbb{R}P^n, TM)$ of bases $\mathfrak{B}$ rather than over $\mathbb{R}P^n$ only, but each class of connections defined by equivalent covariant derivatives are not. These classes are defined only over $\mathbb{R}P^n$.*

*In other words, there is a well-known basis dependence in the definition of the connections, contrary to strict tensor fields. Thus, in particular, a connection form is somehow a "*basis-depending tensor field*" and historically called an '*affinor*' (see Remark 8 and footnote 3.33 and the $\mathfrak{C}^{-1}.\,d\mathfrak{C}$ occurrence depending on the change of bases $\mathfrak{C}$ and which must not appear for a tensor).*

In addition, because $PGL(n+1,\mathbb{R})$ is equivalent to a group quotiented by $\mathbb{R}^*$, two projective connections ω and ω' over $\mathbb{R}P^n$ must be defined as two '*projectively equivalent*' (inhomogeneous horizontal) connections, *i.e.*, $\omega' \sim_P \omega$, whenever we have

$$\omega' \equiv \omega + \mathfrak{r}\mathbb{1}, \tag{3.36}$$

where $\mathbb{1}$ is the identity morphism in TM and $\mathfrak{r}$ is an inhomogeneous horizontal scalar 1-form (over $\mathbb{R}P^n$).

In principle, from the relation (3.35) with $\mathfrak{C} \equiv \lambda\,\mathbb{1}$ and assuming that $\pi' = \lambda\,\pi$ where $\lambda \in \mathcal{O}_{\mathbb{R}P}$, then, the 1-form $\mathfrak{r}$ should be at least exact locally. But, if we assume for instance that $\omega \equiv 0$, then, in the projective frame $\mathfrak{F}$, we obtain that $\nabla'_{\mathfrak{u}}\mathfrak{v}_\alpha = \mathfrak{r}(\mathfrak{u})\mathfrak{v}_\alpha$. This formula means that, whatever is the vector field $\mathfrak{u}$, the vector fields $\mathfrak{v}_\alpha$ are only scaled to $\mathfrak{v}'_\alpha \simeq (1+\mathfrak{r}(\mathfrak{u}))\mathfrak{v}_\alpha$ when passing from a point $p \in \mathbb{R}P^n$ to another point $p' \simeq p + \mathfrak{u}$ in the vicinity of p. But, this kind of change must be related in no way to a significant change of the projective frame field $\mathfrak{F}$ and, consequently, of the projective structure as well. And, moreover, this "insensitivity" or "invariance" must be preserved regardless of whether the 1-form $\mathfrak{r}$ is exact or not.

In addition, this projective equivalence between projective connections is preserved under a change of basis field $\mathfrak{C}$ leaving the representing manifold field $\mathfrak{P}_0$ invariant, up to a scaling, *i.e.*, $\mathfrak{C} \in \Gamma_n(\mathbb{R}^* \times Stab([\xi]))$. Nevertheless, in this case, the projective frame field $\mathfrak{F}$ is changed to a projective frame field $\mathfrak{F}'$ while the vector line $[\mathfrak{v}_0]$ is preserved, *i.e.*, $[\mathfrak{v}'_0] = [\mathfrak{v}_0]$. And therefore, in full generality, we can define the following.

Definition 11. *Let $\mathfrak{u}$ and $\mathfrak{v}$ be any two vector fields in $\chi(\mathbb{R}P^n)$ and $stab([\xi])$ be the Lie* *algebra of* $Stab([\xi]) \simeq Aff(n,\mathbb{R})$*; then, the covariant derivatives* ∇' *and* ∇ *are* 'projectively equivalent,*'* [29] *i.e.,* $\nabla' \sim_P \nabla$ *(or* $\omega' \sim_P \omega$*), if and only if 1)* ∇' *and* ∇ *are equivalent, i.e.,* $\nabla' \sim \nabla$*, and 2) there exists a* $\rho(\mathbb{R}\oplus stab([\xi]))$*-valued horizontal 1-form* $\mathfrak{A}$ *over* $\mathbb{R}P^n$*, i.e.,* $\mathfrak{A} \in \Gamma_n((\mathbb{R}\oplus stab([\xi])) \otimes T^*\mathbb{R}P^n)$*, such that*

$$\nabla'_{\mathfrak{u}}\mathfrak{v} = \nabla_{\mathfrak{u}}\mathfrak{v} + (i_{\mathfrak{u}}\mathfrak{A})\mathfrak{v}, \qquad (or\ \omega' = \omega + \mathfrak{A}) \tag{3.37a}$$

$$\mathfrak{A} = \mathfrak{r}\,\mathbb{1} + \mathfrak{s}\,, \tag{3.37b}$$

where $\mathfrak{s} \in \Gamma_n(stab([\xi]) \otimes T^*\mathbb{R}P^n)$ *is a horizontal* $\rho(stab([\xi]))$*-valued 1-form,* $\mathfrak{r}$ *is a horizontal 1-form over* $\mathbb{R}P^n$ *and* $\mathbb{1}$ *is the identity morphism in* TM*.*

Remark 10. *We can notice that if we require a projective equivalence between* $\nabla_{\mathfrak{u}}\mathfrak{v}$ *and* $\nabla'_{\mathfrak{u}}\mathfrak{v} \equiv e^{-\mathfrak{w}}\mathfrak{C}^{-1}\nabla_{\mathfrak{u}}((e^{\mathfrak{w}}\mathfrak{C})\mathfrak{v})$ *where* $\mathfrak{w} \in \mathcal{O}_{\mathbb{R}P}$ *and* $\mathfrak{C} \in \Gamma_n(Stab([\xi]))$*, then, in particular we obtain, obviously, that* $\mathfrak{r} \equiv d\mathfrak{w}$ *and* $\mathfrak{s} \equiv \mathfrak{C}^{-1}\,.\,d\mathfrak{C} \in \Gamma_n(stab([\xi]) \otimes T^*\mathbb{R}P^n)$*. The element* $\mathfrak{s}$ *is a Maurer-Cartan type 1-form associated with* $stab([\xi])$ *(said also a 'pure gauge' with respect to* $stab([\xi])$ *by physicists). Also, this projective equivalence is obviously preserved under equivalence.*

This equivalence can be used to define traceless projective connections and projective connections when starting from general horizontal inhomogeneous Euclidean connections.

Then, the projective connections are those inhomogeneous Euclidean connections such that the representative manifolds $\mathfrak{P}_0(p)$ remain, somehow, invariants under the infinitesimal actions defined by this given connections, *i.e.*, the representing manifolds $\mathfrak{P}_0(p)$ are parallel transported along the integral manifold defined by the Yano-Ishihara projecting form π. The 1-form π defines a unique decomposition of the tangent spaces in a direct sum of a vertical and a horizontal vector space, *i.e.*, $[\xi]$ and $\mathfrak{P}_0$ respectively.

Additionally, this decomposition gives a structure to the tangent spaces which is similar to a 1-jet bundle structure, *i.e.*, an affine structure of order one inherited from this jet bundle. It also explains the filtration aspect related to a Frobenius foliation of codimension one rather than the graduation one of the associated Grassmann algebra of (co-)tensors.

Then, formally, we have the following definition similar to the one given by Cartan [Car24b]:

[29] Usually, we read in the literature that this equivalence is satisfied if for all vector fields $\mathfrak{u}$ and $\mathfrak{v}$ and 1-form $\mathfrak{r}$ we have $\nabla'_{\mathfrak{u}}\mathfrak{v} \equiv \nabla_{\mathfrak{u}}\mathfrak{v} + (i_{\mathfrak{u}}\,\mathfrak{r})\,\mathfrak{v} + (i_{\mathfrak{v}}\,\mathfrak{r})\,\mathfrak{u}$, but this is true only if ω' and ω are both '*torsion-free*' projective Cartan connections. In general, we have $\nabla'_{\mathfrak{u}}\mathfrak{v} \equiv \nabla_{\mathfrak{u}}\mathfrak{v} + (i_{\mathfrak{u}}\,\mathfrak{r})\,\mathfrak{v}$ if we consider only 1-forms such as $\mathfrak{r}$. See property 4 in the sequel for more details.

Definition 12. *Let* $\mathfrak{F} = \{[\mathfrak{v}_0] \equiv [\xi], \ldots, [\mathfrak{v}_{n+1}]\}$ *be a projective frame and* $\mathfrak{B} = \{\mathfrak{v}_0 \equiv \xi, \ldots, \mathfrak{v}_n\}$ *a corresponding basis where* ξ *is the origin point field of* $\mathfrak{P}_0$*. Then, a projective connection* ω *over* $\mathbb{R}P^n$ *is a* 'pre-projective Cartan connection' *if and only if*

1. $Tr\,\omega_{\mathfrak{B}} = 0$,

2. $\pi(\omega\,.\,\xi) = 0$.

If ∇ is the projective covariant derivative associated with ω, then, in particular, we obtain that

$$\nabla_{\mathfrak{u}}\xi = \mathfrak{q}_0(\nabla_{\mathfrak{u}}\xi), \tag{3.38}$$

where $\mathfrak{u}$ is any vector field in $\chi(\mathbb{R}P^n)$. We deduce also that the relation $\nabla_{\xi}\mathfrak{u} = 0$ is always satisfied.

Also, among these covariant derivatives ∇, we can choose the ones such that the trace of their associated connections ω are vanishing. Indeed, from (3.37), if $Tr(\omega'_{\mathfrak{B}}) \neq 0$ and if ω' and ω are both pre-projective Cartan connections, then, we can take ∇ such that $Tr(\omega_{\mathfrak{B}}) = 0$. It suffices $\mathfrak{A}$ to be such that

$$Tr(\omega_{\mathfrak{B}}) = (n+1)\,\mathfrak{r} + Tr(\mathfrak{s}_{\mathfrak{B}}) + Tr(\omega'_{\mathfrak{B}}) = 0\,. \tag{3.39}$$

Also, the relation (3.38) is, obviously, equivalent to the relation $\mathfrak{p}_0(\nabla_{\mathfrak{u}}\xi) = \pi(i_{\mathfrak{u}}\omega\,.\,\xi)\,\xi = 0$, *i.e.*, $\pi(\nabla_{\mathfrak{u}}\xi) = 0$. We can say that the inhomogeneous vertical field ξ, considered as an affine point in $\mathfrak{P}_0$, remains in the representing manifold $\mathfrak{P}_0$ after any infinitesimal variation defined by ∇. But then, given $\overset{\circ}{\nabla}$ which does not satisfy the condition (3.38), we can obtain from $\overset{\circ}{\nabla}$ another equivalent covariant derivative ∇ satisfying (3.38).

Indeed, let $\overset{\circ}{\nabla}$ and ∇ be two projectively equivalent covariant derivatives defined over $\mathbb{R}P^n$ and satisfying a relation similar to the relation (3.37a) with $\overset{\circ}{\nabla} \equiv \nabla'$ and such that $\pi(\nabla_{\mathfrak{u}}\xi) = 0$. Then, because $\mathfrak{C} \in \Gamma_n(Stab([\xi]))$, we have $(\mathfrak{C}^{-1}\,.\,d\mathfrak{C})\,\xi = 0$, and then, the formula $\pi(\nabla_{\mathfrak{u}}\xi) = 0$ is equivalent to $\pi(\overset{\circ}{\nabla}_{\mathfrak{u}}\xi) = -\mathfrak{h}(\mathfrak{u})$.

This relation defines completely the 1-form $\mathfrak{h}$, and we obtain the pre-projective Cartan connection ω associated with ∇ from the (horizontal) Euclidean connection $\not\omega$ over $\mathbb{R}P^n$ associated with $\not\nabla$ from the relation:

$$\nabla_{\mathfrak{u}}\mathfrak{v} = \not\nabla_{\mathfrak{u}}\mathfrak{v} - \pi(\not\nabla_{\mathfrak{u}}\xi)\mathfrak{v}, \tag{3.40}$$

for any vector fields $\mathfrak{u}$ and $\mathfrak{v}$ in $\chi(\mathbb{R}P^n)$. Actually, it follows that $\not\nabla$ is also a pre-projective covariant derivative, but not a pre-projective Cartan covariant derivative requiring $\pi(\not\nabla_{\mathfrak{u}}\xi) = 0$.

The relation (3.40) exhibits a true projective behavior under scaling. Indeed, considering a scaling on TM and T^*M such that any vector field $\mathfrak{v}$ and any 1-form σ are transformed into, respectively, the vector field $\lambda\,\mathfrak{v}$ and the 1-form $\lambda^{-1}\,\sigma$ where λ is any smooth function on M, then, $\nabla\mathfrak{v}$ is transformed into $\not\nabla(\lambda\,\mathfrak{v}) - \lambda^{-1}\,\pi(\not\nabla(\lambda\,\xi))\lambda\,\mathfrak{v}$. And thus, we have for any vector field $\mathfrak{w}$:

$$\begin{aligned}
\not\nabla_{\mathfrak{w}}(\lambda\,\mathfrak{v}) - \lambda^{-1}\,\pi(\not\nabla_{\mathfrak{w}}(\lambda\,\xi))\lambda\,\mathfrak{v} &= (i_{\mathfrak{w}}d\lambda)\mathfrak{v} + \lambda\,\not\nabla_{\mathfrak{w}}(\mathfrak{v}) - \pi(\not\nabla_{\mathfrak{w}}(\lambda\,\xi))\mathfrak{v}\\
&= (i_{\mathfrak{w}}d\lambda)\mathfrak{v} + \lambda\,\not\nabla_{\mathfrak{w}}(\mathfrak{v}) - \pi((i_{\mathfrak{w}}d\lambda)\xi + \lambda\,\not\nabla_{\mathfrak{w}}(\xi))\mathfrak{v}\\
&= (i_{\mathfrak{w}}d\lambda)\mathfrak{v} + \lambda\,\not\nabla_{\mathfrak{w}}(\mathfrak{v}) - (i_{\mathfrak{w}}d\lambda)\mathfrak{v} - \lambda\,\pi(\not\nabla_{\mathfrak{w}}(\xi))\mathfrak{v}\\
&= \lambda\left(\not\nabla_{\mathfrak{w}}(\mathfrak{v}) - \pi(\not\nabla_{\mathfrak{w}}(\xi))\mathfrak{v}\right).
\end{aligned}$$

Hence, $\nabla\mathfrak{v}$ is correctly transformed into $\lambda\,\nabla\mathfrak{v}$ on TM.

There would be another way to obtain a projective connection ω from a horizontal Euclidean connection $\not\omega$ over $\mathbb{R}P^n$, in particular if there are not projectively equivalent. A priori, it would be just sufficient to pose for any vector fields $\mathfrak{u}$ and $\mathfrak{v}$ in $\chi(\mathbb{R}P^n)$ the following definition of the projective covariant derivative ∇ from $\not\nabla$ defined *on* $\mathbb{R}P^n$:

$$\nabla_{\mathfrak{u}}\mathfrak{v} = \mathfrak{q}_0(\not\nabla_{\mathfrak{u}}\mathfrak{v})\,. \tag{3.41}$$

But, the covariant derivative ∇ becomes not covariant on $\mathbb{R}^{n+1}$ with respect to $GL(n+1,\mathbb{R})$ because the projector $\mathfrak{q}_0$ somehow "breaks" the $GL(n+1,\mathbb{R})$ covariance which could be inherited from $\not\nabla$. Indeed, this covariance means that the equivalence relation $\nabla \sim \nabla'$ defined by (3.31a) must be preserved, *i.e.*, if the two following equivalence relations $\nabla \sim \nabla'$ and $\not\nabla \sim \not\nabla'$ are satisfied both together for the same common element $\mathfrak{C} \in \Gamma_n(GL(n+1,\mathbb{R}))$, then, if $\nabla_{\mathfrak{u}}\mathfrak{v} = \mathfrak{q}_0(\not\nabla_{\mathfrak{u}}\mathfrak{v})$, we must have also $\nabla'_{\mathfrak{u}}\mathfrak{v} = \mathfrak{q}_0(\not\nabla'_{\mathfrak{u}}\mathfrak{v})$ for any vector fields $\mathfrak{u}$ and $\mathfrak{v}$.

Therefore, we would obtain $\mathfrak{C}^{-1}(\nabla_{\mathfrak{u}}(\mathfrak{C}(\mathfrak{v}))) = \mathfrak{q}_0(\mathfrak{C}^{-1}(\not\nabla_{\mathfrak{u}}(\mathfrak{C}(\mathfrak{v}))))$ or, equivalently, $\mathfrak{C}^{-1}(\mathfrak{q}_0(\not\nabla_{\mathfrak{u}}(\mathfrak{C}(\mathfrak{v})))) = \mathfrak{q}_0(\mathfrak{C}^{-1}(\not\nabla_{\mathfrak{u}}(\mathfrak{C}(\mathfrak{v}))))$. Thus, $\mathfrak{C}$ must be such that $\mathfrak{C}^{-1}\circ\mathfrak{q}_0 = \mathfrak{q}_0\circ\mathfrak{C}^{-1}$, and then, $\mathfrak{p}_0\circ\mathfrak{C}^{-1}\circ\mathfrak{q}_0 = 0$. But, we have also $\mathfrak{C}^{-1}(\nabla_{\mathfrak{u}}(\mathfrak{C}(\mathfrak{v}))) = \mathfrak{q}_0(\mathfrak{C}^{-1}\,\mathfrak{p}_0(\not\nabla_{\mathfrak{u}}(\mathfrak{C}(\mathfrak{v})))) + \mathfrak{q}_0(\mathfrak{C}^{-1}\,\mathfrak{q}_0(\not\nabla_{\mathfrak{u}}(\mathfrak{C}(\mathfrak{v}))))$.

Hence, we deduce that $\mathfrak{q}_0\circ\mathfrak{C}^{-1}\circ\mathfrak{p}_0 = 0$, and then $\mathfrak{C}^{-1}\circ\mathfrak{p}_0 = \mathfrak{p}_0\circ\mathfrak{C}^{-1}$. Therefore, $\mathfrak{C}$ must be completely reducible and $\mathfrak{C}\in\Gamma_n(\mathbb{R}^*\times(\{1\}\oplus GL(n,\mathbb{R})))$. But, $GL(n,\mathbb{R})$ is an "admissible" group of covariance since it is a normal subgroup of the structural group $\mathit{Aff}(n,\mathbb{R})$.

And then, we conclude that the projective covariant derivative ∇ defined by the relation (3.41) remains projective under the adjoint action of $\mathbb{R}^*\times(\{1\}\oplus GL(n,\mathbb{R}))$ and, as foreseen, that $GL(n+1,\mathbb{R})$ is not the suitable group of "projective" covariance.

As a consequence, the relation (3.41), but actually behind, the general definition (3.38) for a projective derivative (with the invariance of the representing manifold field $\mathfrak{P}_0$) is only preserved under the adjoint action of $\mathbb{R}^*\times(\{1\}\oplus GL(n,\mathbb{R}))$, and therefore, it is not a covariant derivative with respect to changes of projective frames. But, another feature can be added to the definition (3.38) to obtain somehow "normal forms" associated with any projective covariant derivative as indicated in the following.

3. General properties

Theorem 4. *Let $\widetilde{\nabla}$ be any projective covariant derivative such that the $\mathcal{O}_{\mathbb{R}P}$-linear map $\widetilde{\nabla}\xi : \mathfrak{u}\in\chi(\mathbb{R}P^n)\to\widetilde{\nabla}_{\mathfrak{u}}\xi\in\chi(\mathbb{R}P^n)$ is such that $rk(\widetilde{\nabla}\xi) = n$ and $\widetilde{\nabla}_{\mathfrak{u}}\tilde{\xi}\neq\tilde{\mathfrak{q}}_0(\mathfrak{u})$; Then, after a change of projective frame by an element $\mathfrak{C}\in\Gamma_n(GL(n+1,\mathbb{R}))$, we can find a projective covariant derivative ∇ such that*

1. $(\nabla,\pi)\sim(\widetilde{\nabla},\tilde{\pi})$, *and*

2. $\nabla_{\mathfrak{u}}\xi = \mathfrak{q}_0(\mathfrak{u})\equiv\underline{\mathfrak{u}}$ *for any vector field* $\mathfrak{u}\in\chi(\mathbb{R}P^n)$.

Proof. Indeed, if $\widetilde{\nabla}$ is a projective covariant derivative, then, in particular, the $\mathcal{O}_{\mathbb{R}P}$-morphism $\widetilde{\nabla}\tilde{\xi} : \mathfrak{u}\in\Gamma_n(TM)\longrightarrow\widetilde{\nabla}_{\mathfrak{u}}\tilde{\xi}\in\Gamma_n(TM)$ is such that $rk(\widetilde{\nabla}\tilde{\xi})\equiv r\leq n$ because $\tilde{\mathfrak{p}}_0(\widetilde{\nabla}_{\mathfrak{u}}\tilde{\xi}) = 0$.

Therefore, if $r=n$, and because $\Gamma_n(GL(n+1,\mathbb{R}))$ is a projective module, then there exists an element $\widetilde{\mathfrak{C}}\in\Gamma_n(GL(n+1,\mathbb{R}))$ factorizing $\widetilde{\nabla}\tilde{\xi}$, *i.e.*, $\widetilde{\mathfrak{C}}$ is such that $\widetilde{\nabla}\tilde{\xi}=\widetilde{\mathfrak{C}}\circ\tilde{\mathfrak{q}}_0$ (and not $\tilde{\mathfrak{q}}_0\circ\widetilde{\mathfrak{C}}$ because $\widetilde{\nabla}\tilde{\xi}$ must be horizontal), or, equivalently, $\widetilde{\nabla}_{\mathfrak{u}}\tilde{\xi}=\widetilde{\mathfrak{C}}(\mathfrak{u}-\tilde{\pi}(\mathfrak{u})\,\tilde{\xi})$ for any vector field $\mathfrak{u}$ on $\mathbb{R}P^n$. But, because $\tilde{\mathfrak{p}}_0(\widetilde{\nabla}_u\tilde{\xi})=0$, then there exists a (non-unique) vector field $\mathfrak{v}$ we fix such that $\widetilde{\mathfrak{C}}(\mathfrak{u}-\tilde{\pi}(\mathfrak{u})\,\tilde{\xi})=\mathfrak{v}-\tilde{\pi}(\mathfrak{v})\,\tilde{\xi}$, or, equivalently, $\widetilde{\mathfrak{C}}(\underline{\mathfrak{u}})=\underline{\mathfrak{v}}$. Then, because $GL(n+1,\mathbb{R})$ acts transitively on each tangent space T_pM, there exists an element $\mathfrak{C}\in\Gamma_n(GL(n+1,\mathbb{R}))$ such that $\mathfrak{v}=\mathfrak{C}(\mathfrak{u})$. Moreover, $\mathfrak{C}$ is independent on $\mathfrak{u}$ since it can be defined completely from all of the relations $\mathfrak{v}_\alpha=\mathfrak{C}(\mathfrak{u}_\alpha)$ and $\widetilde{\mathfrak{C}}(\mathfrak{u}_\alpha-\tilde{\pi}(\mathfrak{u}_\alpha)\,\tilde{\xi})=\mathfrak{v}_\alpha-\tilde{\pi}(\mathfrak{v}_\alpha)\,\tilde{\xi}$ obtained for all of the basis vector fields $\mathfrak{u}_\alpha$ of any given basis of $\Gamma_n(TM)$ given both with their corresponding vector fields $\mathfrak{v}_\alpha$ given partly by $\widetilde{\nabla}_{\mathfrak{u}_\alpha}\tilde{\xi}$.

Therefore, if we define the inhomogeneous 1-form π by the relation $\pi(\mathfrak{w})\equiv\tilde{\pi}(\mathfrak{C}(\mathfrak{w}))$ for any vector field $\mathfrak{w}$, then $\pi(\xi)=1$ where $\xi\equiv\mathfrak{C}^{-1}(\tilde{\xi})$. Thus, we obtain $\widetilde{\nabla}_{\mathfrak{u}}\tilde{\xi}=\mathfrak{C}(\mathfrak{u}-\pi(\mathfrak{u})\,\xi)$. Hence, we have $\mathfrak{C}^{-1}(\widetilde{\nabla}_{\mathfrak{u}}\tilde{\xi})=\mathfrak{C}^{-1}(\widetilde{\nabla}_{\mathfrak{u}}\mathfrak{C}(\xi))=\mathfrak{u}-\pi(\mathfrak{u})\,\xi\equiv\nabla_{\mathfrak{u}}\xi$. Moreover, the map $\mathfrak{C}^{-1}\circ\widetilde{\mathfrak{C}}$ is a bijective map between the horizontal spaces, *i.e.*, we have $\mathfrak{C}^{-1}\circ\widetilde{\mathfrak{C}}\circ\tilde{\mathfrak{q}}_0=\mathfrak{q}_0$, and additionally, $\widetilde{\mathfrak{C}}$ is only defined partially from the relation $\widetilde{\nabla}\tilde{\xi}=\widetilde{\mathfrak{C}}\circ\tilde{\mathfrak{q}}_0$. Indeed, $\widetilde{\mathfrak{C}}$ is fully defined only on the horizontal spaces from the previous relation $\widetilde{\mathfrak{C}}(\underline{\mathfrak{u}})=\underline{\mathfrak{v}}$. But, nevertheless, we can completely define $\widetilde{\mathfrak{C}}$ given a nonvanishing germ of function $\tilde{\mu}$ at any point p such that $\mathfrak{C}^{-1}\circ\widetilde{\mathfrak{C}}(\tilde{\xi})=\tilde{\mu}\,\xi$.

Lastly, if we assume $\widetilde{\nabla}_{\mathfrak{u}}\tilde{\xi}=\mathfrak{C}(\mathfrak{u}-\pi(\mathfrak{u})\,\xi)\equiv\tilde{\mathfrak{q}}_0(\mathfrak{u})=\mathfrak{u}-\tilde{\pi}(\mathfrak{u})\,\tilde{\xi}$, *i.e.*, $\mathfrak{C}(\mathfrak{u})\equiv\mathfrak{u}+(\tilde{\pi}(\mathfrak{C}(\mathfrak{u}))-\tilde{\pi}(\mathfrak{u}))\,\tilde{\xi}$, then, $\ker\mathfrak{C}=\{r\,\tilde{\xi};\,r\in\mathbb{R}\}\neq 0$, *i.e.*, $\mathfrak{C}$ is not bijective which is not possible. And therefore, wo obtain $\widetilde{\mathfrak{C}}\neq\mathbb{1}$. □

Also, considering that we have locally the soldering $\mathcal{J}_0\simeq_{loc.}\mathbb{R}P^n$ and $P_0\simeq_{loc.}\mathcal{J}_0\simeq_{loc.}\mathbb{R}^n$, we deduce the following.

Lemma 2. *If the vector field $\xi:p\in\mathbb{R}P^n\longrightarrow\xi(p)\in T_pM$ is an* embedding, *then the rank of the linear map $\nabla\xi:u\in\Gamma_n(TM)\longrightarrow\nabla_u\xi\in\Gamma_n(TM)$ is maximal, i.e., it is equal to n.*

Proof. Indeed, the tangent map $T\xi$ is nothing more than the *symbol map* of $\nabla\xi$ and the maximal rank of the map $u\longrightarrow i_u\omega$ is n because the projective connection ω is horizontal. In particular, we can identify $p\in\mathcal{J}_0\simeq_{loc.}P_0\subset\mathbb{R}^{n+1}$ with the origin s_p of $\mathfrak{P}_0(p)$ such that $\overrightarrow{op}=\overrightarrow{ps_p}=\xi(p)$, and then for all $p\in P_0$ we have $P_0\simeq\mathfrak{P}_0(p)$.

This is the very natural choice historically made by É. Cartan meaning that he considered that the field of affine hyperplanes $\mathfrak{P}_0$ is constant and that π is a constant 1-form; And thus, that π is exact, and then, equal to the differential of a Cartesian coordinate in $\mathbb{R}^{n+1}$. $\square$

In conclusion, we have the following theorem:

Theorem 5. *Let $\mathfrak{u}$ and $\mathfrak{v}$ be any vector fields over $\mathbb{R}P^n$ (i.e., in $\chi(\mathbb{R}P^n)$); Then, any projective covariant derivative defined over $\mathbb{R}P^n$ is equivalent to a pre-projective covariant derivative ∇ with associated pre-projective Cartan connection ω (Def. 12) such that*

1. *there exists a horizontal Euclidean covariant derivative $\overset{\star}{\nabla}$ over $\mathbb{R}P^n$ such that $\nabla_{\mathfrak{u}}\mathfrak{v} \equiv \overset{\star}{\nabla}_{\mathfrak{u}}\mathfrak{v} - \pi(\overset{\star}{\nabla}_{\mathfrak{u}}\xi)\mathfrak{v}$ (see (3.40)),*

2. *if ξ is an embedding from $\mathbb{R}P^n$ to TM, then we have $\nabla_{\mathfrak{u}}\xi = \mathfrak{q}_0(\mathfrak{u})$.*

Then, we have the following definitions.

Definition 13. *We call* projective Cartan connection ω *any pre-projective Cartan connection ω verifying the properties 1 and 2 of the theorem 5. Moreover, given a basis field $\mathfrak{B} = \{\mathfrak{v}_0, \ldots, \mathfrak{v}_n\}$, ω is a* $\mathfrak{B}$-complete *projective Cartan connection, if, in addition, for all $i = 1, \ldots, n$, then the n scalar 1-forms $\mathfrak{v}^{*,i}(\omega\,.\,\xi)$ are $\mathcal{O}_{\mathbb{R}P}$-linearly independent horizontal* closed *1-forms over $\mathbb{R}P^n$.*

The last condition in this definition can always be satisfied. Indeed, considering a projective Cartan connection ω and a projectively equivalent projective Cartan connection ω', and thus, such that $\omega' = \omega + \mathfrak{A}$ and $\nabla\xi = \nabla'\xi = \mathfrak{q}_0$, then, there always exists a $\mathfrak{A}$ such that ω' satisfies the last condition of the definition 13. For, the 1-form $\mathfrak{A}$ must be such that ${\omega'}^i_{\mathfrak{B},0} = \mathfrak{v}^{*,i}(\omega\,.\,\xi) + \mathfrak{v}^{*,i}(\mathfrak{A}\,.\,\xi) \equiv \omega^i_{\mathfrak{B},0} + \mathfrak{A}^i_{\mathfrak{B},0}$ is locally exact. But, the n scalar forms $\mathfrak{A}^i_{\mathfrak{B},0}$ are inhomogeneous and horizontal, and thus, the n-dimensional Pfaff system $\{{\omega'}^i_{\mathfrak{B},0}\}_{i=1,\ldots,n}$ is completely integrable over the n-dimensional manifold $\mathbb{R}P^n$. Therefore, from the Frobenius theorem, $\mathfrak{A}$ exists such that ω' satisfies the condition *3*.

4. The projective (co-)tensors versus the Euclidean (co-)tensors – General remarks.

We have seen that the general definition (3.38) of projective derivative is only covariant with respect to the adjoint action of $\mathbb{R}^*\times(\{1\}\oplus GL(n,\mathbb{R}))\subset\mathbb{R}^*\times Stab([\xi])$. This covariance is shared by any projective (co-)tensor field. Indeed, the following unique projective decomposition: $\mathfrak{u}=\mathfrak{u}_0\,\xi+\underline{\mathfrak{u}}\in\chi(\mathbb{R}P^n)$ where $\mathfrak{u}_0\equiv\pi(\mathfrak{u})$, which generalizes the decomposition (3.6) can be interpreted as the *reduction* of the vector field $\mathfrak{u}$ with respect to the linear group $GL(n,\mathbb{R})$ and it will be easy to make similar reduction on any (co-)tensor of arbitrary rank. But also, this reduction process does not commute with changes of frames defined by the more general adjoint group action of $\mathbb{R}^*\times Stab([\xi])$ (or $GL(n+1,\mathbb{R})$). Moreover, this projective decomposition may not be preserved applying a projective derivative ∇, *i.e.*, $\mathfrak{q}_0\circ\nabla\neq\nabla\circ\mathfrak{q}_0$.

If we consider a change of projective structure over $\mathbb{R}P^n$ via the equivalence defined by the relations (3.31) and with some elements $\mathfrak{C}$ in $GL(n+1,\mathbb{R})/(\mathbb{R}^*\times Stab([\xi]))\simeq PGL(n+1,\mathbb{R})$, then this reduction process is the same as for the so-called Euclidean (co-)tensors. For, we have, as usual, two methods: first, we can reduce any tensor T with respect to the two homomorphic normal subgroups $GL(n,\mathbb{R})$ of $Stab([\xi])$ and $Stab([\xi'])$ where $\xi'\equiv\mathfrak{C}(\xi)$ to obtain two sets of $GL(n,\mathbb{R})$-irreducible (co-)tensors T_{ir} and T'_{ir} associated with T, and then, each of them is expressed linearly relative to the other, or, second, we transform T into T' via $\mathfrak{C}$ and we reduce T' with respect to the $GL(n,\mathbb{R})$ structure defined by ξ'.

But, if $\mathfrak{C}\in Stab([\xi])/(\{1\}\oplus GL(n,\mathbb{R}))$, *i.e.*, $\mathfrak{C}$ is in the translation subgroup $T([\xi])$ of $Stab([\xi])$ and the projective structure over $\mathbb{R}P^n$ is not modified, we are faced to a tensorial situation with no equivalent in Euclidean geometry: there exist $GL(n+1,\mathbb{R})/GL(n,\mathbb{R})$ transformations for which all of the (co-)tensors transforms as *(co-)affinors*, *i.e.*, the group actions on the (co-)tensors T are *not linear but affine* on some nonvanishing $GL(n,\mathbb{R})$-irreducible components of the T's. In other words, some $GL(n,\mathbb{R})$-irreducible components of the T's are mixed under the affine transformations. As a result, some interior products (contractions) of some $GL(n,\mathbb{R})$-irreducible (co-)tensors with respect to some other $GL(n,\mathbb{R})$-irreducible (co-)tensors will not be covariant with respect to $Stab([\xi])$.

More specifically, for any vector field $\mathfrak{u}\in\chi(\mathbb{R}P^n)$ and 1-form $\alpha\in\Gamma_n(T^*M)$, we have the unique following vertical and horizontal decompositions:

$$\mathfrak{u} = \mathfrak{u}_0\,\xi + \underline{\mathfrak{u}}\,, \qquad \mathfrak{u}_0 = \pi(\mathfrak{u})\,, \qquad \pi(\underline{\mathfrak{u}}) = 0\,, \tag{3.42a}$$

$$\alpha = \alpha_0\,\pi + \underline{\alpha}\,, \qquad \alpha_0 = i_\xi\alpha\,, \qquad \underline{\alpha}(\xi) = 0\,. \tag{3.42b}$$

Then, let $\mathfrak{C}$ be such that $\mathfrak{C} \in \Gamma_n(\mathbb{R}^*{\times}Stab([\xi]))$, we obtain that

$$\begin{pmatrix} \underline{\mathfrak{u}}' \\ u_0' \end{pmatrix} = \begin{pmatrix} \underline{\mathfrak{C}} & 0 \\ \underline{\mathfrak{c}} & \mathfrak{c}_0 \end{pmatrix} \begin{pmatrix} \underline{\mathfrak{u}} \\ u_0 \end{pmatrix} \iff \begin{cases} \underline{\mathfrak{u}}' = \underline{\mathfrak{C}}(\underline{\mathfrak{u}})\,, \\ u_0' = \underline{\mathfrak{c}}(\underline{\mathfrak{u}}) + \mathfrak{c}_0\,\mathfrak{u}_0 \equiv \mathfrak{c}(\mathfrak{u})\,, \end{cases} \tag{3.43}$$

where $\mathfrak{c} \equiv \pi \circ \mathfrak{C} \equiv {}^t\mathfrak{C}(\pi) \equiv \underline{\mathfrak{c}} + \mathfrak{c}_0\,\pi$, $\mathfrak{c}_0 \equiv \mathfrak{c}(\xi) \in \mathcal{O}^*_{\mathbb{R}P}$, $\underline{\mathfrak{c}}(\xi) = 0$ and $\mathfrak{c}_0 \neq 0$. Moreover, from the equality: $\alpha(\mathfrak{u}) = \alpha'(\mathfrak{u}') \iff \underline{\alpha}(\underline{\mathfrak{u}}) + \alpha_0\,\mathfrak{u}_0 = \underline{\alpha}'(\underline{\mathfrak{u}}') + \alpha_0'\,\mathfrak{u}_0'$, and denoting by $\mathfrak{C}^\dagger \equiv {}^t\mathfrak{C}^{-1}$ the contragredient representation of $\mathfrak{C}$, we deduce also that

$$\begin{pmatrix} \underline{\alpha}' \\ \alpha_0' \end{pmatrix} = \begin{pmatrix} \underline{\mathfrak{C}}^\dagger & -\underline{\mathfrak{c}}^\dagger \\ 0 & \mathfrak{c}_0^{-1} \end{pmatrix} \begin{pmatrix} \underline{\alpha} \\ \alpha_0 \end{pmatrix} \iff \begin{cases} \underline{\alpha}' = \underline{\mathfrak{C}}^\dagger(\underline{\alpha}) - \alpha_0\,\underline{\mathfrak{c}}^\dagger\,, \\ \alpha_0' = \alpha_0\,\mathfrak{c}_0^{-1}\,, \end{cases} \tag{3.44}$$

where $\underline{\mathfrak{c}}^\dagger \equiv -\mathfrak{c}_0^{-1}\,\underline{\mathfrak{C}}^\dagger(\underline{\mathfrak{c}})$ and $\underline{\mathfrak{c}}^\dagger(\xi) = 0$. Therefore, obviously, we have also the following relations:

$$\mathfrak{u}' = \underline{\mathfrak{C}}(\underline{\mathfrak{u}}) + \mathfrak{c}(\mathfrak{u})\,\xi\,, \tag{3.45a}$$

$$\alpha' = \underline{\mathfrak{C}}^\dagger(\underline{\alpha}) - (i_\xi\alpha)\,\mathfrak{c}^\dagger\,, \tag{3.45b}$$

where $\mathfrak{c}^\dagger \equiv \underline{\mathfrak{c}}^\dagger - \mathfrak{c}_0^{-1}\,\pi$. In particular, we deduce also that

$$\mathfrak{u} = \underline{\mathfrak{C}}^{-1}(\underline{\mathfrak{u}}') + \mathfrak{c}^{-1}(\mathfrak{u}')\,\xi\,, \tag{3.46}$$

where $\mathfrak{c}^{-1} \equiv \mathfrak{c}_0^{-1}(\pi - \underline{\mathfrak{C}}^\dagger(\underline{\mathfrak{c}}))$. Hence, from (3.43) and (3.44), we see that only tensorial homogeneous expressions of the following general form:

$$(\underline{\alpha}_1 \otimes \ldots \otimes \underline{\alpha}_p) \otimes (\xi \otimes \xi \otimes \ldots \otimes \xi)\,, \tag{3.47}$$

where the $\underline{\alpha}_i$ are horizontal forms such that $\underline{\alpha}_i(\xi) = 0$, are (co-)tensors or mixed tensors with respect to $\mathbb{R}^*{\times}Stab([\xi])$. Actually, the action of this group is no more effective, and then, only the subgroup $\mathbb{R}^*{\times}(\{1\}{\oplus}GL(n,\mathbb{R})) \subset \mathbb{R}^*{\times}Stab([\xi])$ acts on the tensorial terms (3.47).

But, no (co-)tensors or mixed tensors remain as such after derivation by a projective covariant derivative, *i.e.*, the tensorial property is not stable under projective covariant derivation. This "instability" yields to "filtered prolongations" rather than "graded prolongations" of (co-)tensor fields.

Indeed, for instance, let ∇ be a projective covariant derivative, then, we obtain that

$$\begin{aligned} Ad(\mathfrak{C}^{-1})(\nabla_{\mathfrak{u}})\mathfrak{v} &= \nabla_{\mathfrak{u}}\mathfrak{v} + (\mathfrak{C}^{-1}\, i_{\mathfrak{u}} d\mathfrak{C})\mathfrak{v} \\ &= \nabla_{\mathfrak{u}}\mathfrak{v} + \left(\underline{\mathfrak{C}}^{-1}\, i_{\mathfrak{u}} d\underline{\mathfrak{C}}\right)\underline{\mathfrak{v}} + (\mathfrak{c}_0^{-1} i_{\mathfrak{u}} d\mathfrak{c}_0)\mathfrak{v}_0\,\xi + \mathfrak{c}_0^{-1}\left(i_{\mathfrak{u}} d\underline{\mathfrak{c}}(\underline{\mathfrak{v}}) - \underline{\mathfrak{c}}\Big(\left(\underline{\mathfrak{C}}^{-1}\, i_{\mathfrak{u}} d\underline{\mathfrak{C}}\right)\underline{\mathfrak{v}}\Big)\right)\xi \\ &= \nabla_{\mathfrak{u}}\mathfrak{v} + \left(\widehat{\mathfrak{C}}^{-1}\, i_{\mathfrak{u}} d\widehat{\mathfrak{C}}\right)\mathfrak{v} + \mathfrak{r}(\mathfrak{u},\underline{\mathfrak{v}})\,\xi \\ &\equiv \nabla'_{\mathfrak{u}}\mathfrak{v} + \mathfrak{r}(\mathfrak{u},\underline{\mathfrak{v}})\,\xi\,, \end{aligned} \tag{3.48}$$

where $\mathfrak{r}(\mathfrak{u},\underline{\mathfrak{v}}) \equiv \mathfrak{c}_0^{-1}\left(i_{\mathfrak{u}} d\underline{\mathfrak{c}}(\underline{\mathfrak{v}}) - \underline{\mathfrak{c}}\left(\left(\underline{\mathfrak{C}}^{-1}\, i_{\mathfrak{u}} d\underline{\mathfrak{C}}\right)\underline{\mathfrak{v}}\right)\right)$ and $\mathfrak{r}(\mathfrak{u},\underline{\mathfrak{v}}) \equiv \mathfrak{r}(\underline{\mathfrak{u}},\underline{\mathfrak{v}})$ because we have horizontal forms only, and $\widehat{\mathfrak{C}}(\mathfrak{v}) \equiv \underline{\mathfrak{C}}(\underline{\mathfrak{v}}) + \mathfrak{c}_0\mathfrak{v}_0$ and $\widehat{\mathfrak{C}} \in \mathbb{R}^*\times Stab([\xi])$. Moreover, we have that $Ad(\mathfrak{C}^{-1})(\nabla) \sim_P \nabla$ and $\nabla' \sim_P \nabla$. Hence, assuming $\mathfrak{v} \equiv \underline{\mathfrak{v}}$, then, if one of the three derivatives of $\mathfrak{v}$ is horizontal, *e.g.*, $\mathfrak{q}_0(\nabla_{\mathfrak{u}}\underline{\mathfrak{v}}) = \underline{\nabla_{\mathfrak{u}}\underline{\mathfrak{v}}} = \nabla_{\mathfrak{u}}\underline{\mathfrak{v}}$, then, necessarily, due to the occurrence of the term $\mathfrak{r}(\mathfrak{u},\underline{\mathfrak{v}})\,\xi$ in (3.48), one of the two other covariant derivatives cannot be horizontal. In other words, the "horizontality" of any covariant derivative of $\underline{\mathfrak{v}}$ is projective frame dependent, and therefore, $\nabla_{\mathfrak{u}}\underline{\mathfrak{v}}$ is not an horizontal tensor.

Similarly, "verticality" would be preserved if no covariant derivative of ξ exists such that $\nabla_{\mathfrak{u}}\xi$ is horizontal; a property which cannot be obtained from the relation 2 in theorem 4. Hence, $\nabla_{\mathfrak{u}}\mathfrak{v}$ is not a tensor over $\mathbb{R}P^n$ with respect to $Stab([\xi])$. It defines rather a "filtered prolongation" of $\mathfrak{v}$ which is an element of $\Gamma_n(J_1(TM))$ where $J_1(TM)$ is the 1-jet bundle of vector fields in $\chi(M)$ over $\mathbb{R}P^n$. Then, we can define a first prolongation map j^1 such that:

$$\mathfrak{v} \in \chi(\mathbb{R}P^n) \xrightarrow{\;j^1\;} \begin{pmatrix} \nabla\mathfrak{v} \\ \mathfrak{v} \end{pmatrix} \in \Gamma_n(J_1(TM)) \xrightarrow{\;\mathfrak{C}^1\;} \begin{pmatrix} \nabla'\mathfrak{v} + \xi\otimes\mathfrak{r}_{\underline{\mathfrak{v}}} \\ \mathfrak{v} \end{pmatrix} \in \Gamma_n(J_1(TM))\,, \tag{3.49}$$

where $\mathfrak{r}_{\underline{\mathfrak{v}}}$ is the 1-form such that $\mathfrak{r}_{\underline{\mathfrak{v}}}(\mathfrak{u}) \equiv \mathfrak{r}(\mathfrak{u},\underline{\mathfrak{v}})$, and where $\mathfrak{C}^1$ is defined from the adjoint action of $\mathfrak{C}$ indicated above in relation (3.48). The grading is defined in such a way that $\mathfrak{v}$ is the 0-component of $j^1(\mathfrak{v})$. The action of linear groups such as $GL(n+1,\mathbb{R})$ on $J_1(TM)$ is graded only for true (co-)tensors, *i.e.*, Euclidean (co-)tensors, and the affine aspect of $j^1(\mathfrak{v})$ is unveiled by the dependence with respect to the term $\mathfrak{r}_{\underline{\mathfrak{v}}}$ in $\mathfrak{C}^1\circ j^1(\mathfrak{v})$ after the adjoint action of $\mathfrak{C}$.

We will not go further on the general theory of such affinors, nor on those affine aspects related to the horizontal/vertical splitting. Situations where such aspects appear, also few in the results presented here, will be handled on a case by case basis. Let us add that such aspects are very usual in relativity theory because the present horizontal/vertical splitting corresponds structurally exactly to the space/time splitting of this theory. Moreover, when using the terms tensors or co-tensors in the sequel, we will always refer to their tensorial aspects with respect to the group $GL(n+1,\mathbb{R})$ only. All other terms such as (co-)affinors will be used only in particular cases for pointing out the specific structural aspect of the tangent space $T\mathbb{R}P^n$.

Now, additionally, we extend the definition of the projective covariant derivative ∇ to differential forms. Thus, let θ, $\mathfrak{u}$ and $\mathfrak{v}$ be respectively any 1-form and any two vector fields over $\mathbb{R}P^n$. Then, the action of ∇ on differential 1-forms is defined from the "usual" relation: $i_{\mathfrak{u}}(\nabla_{\mathfrak{v}}\theta) \equiv i_{\mathfrak{v}}\, d(\theta(\mathfrak{u})) - \theta(\nabla_{\mathfrak{v}}\mathfrak{u})$. Hence, if $\mathfrak{F}^* = \{[\mathfrak{v}^{*,0}], \ldots, [\mathfrak{v}^{*,n+1}]\}$ is the dual projective coframe of dual 1-forms of $\mathfrak{F}$, and thus such that $\mathfrak{v}^{*,\alpha}(\mathfrak{v}_\beta) \equiv \delta^\alpha_\beta$, then we have that

$$\nabla \mathfrak{v}^{*,\alpha} = -\mathfrak{v}^{*,\beta} \otimes \omega^\alpha_\beta\,. \tag{3.50}$$

Thus, we deduce also the "obvious" following result with the same notations as in theorem 5:

Corollary 1. *The dual projective connection ω^* is such that $\omega^* = -{}^t\omega$, where ${}^t\omega$ is the transpose of ω, and $\nabla_{\mathfrak{u}}\sigma = \overset{\circ}{\nabla}_{\mathfrak{u}}\sigma + \pi(\overset{\circ}{\nabla}_{\mathfrak{u}}\xi)\sigma$ for any vector field $\mathfrak{u}$ in $\chi(\mathbb{R}P^n)$ and any 1-form σ in $\Gamma_n(T^*M)$.*

Futhermore, we denote also by ∇ the '*extended*' projective covariant derivative which acts on mixed tensor fields, *i.e.*, on the $\mathcal{O}_{\mathbb{R}P}$-algebra $\bigoplus_{k,h=0}^{\infty}\Big((TM)^{\otimes k} \otimes \bigwedge^h T^*M\Big)$ of tensor products of differential h-forms and tensors fields of order k and such that its restriction on the ring of vector fields and of differential 1-forms is ∇. Lastly, we can notice that whatever are $\mathfrak{v}$ and θ defined as above, a simple computation gives the following result: $\nabla(\mathfrak{v}\otimes\theta) = \nabla\mathfrak{v}\otimes\theta + \mathfrak{v}\otimes\nabla\theta$ $= \overset{\circ}{\nabla}(\mathfrak{v}\otimes\theta)$ where $\overset{\circ}{\nabla}$ satisfies the relation 1 in the theorem 5.

5. *A fundamental exemple on $\mathbb{R}P^n$: the Cartan approach (1924).*

In this case, we start with the vector space $M \equiv \mathbb{R}^{n+1}$ with local homogeneous Euclidean coordinates $\tau_0, \ldots, \tau_n$. Thus, the affine space P_0 is the space of points $p \in \mathbb{R}^{n+1}$ such that $\tau_0 = 1$ and the projecting form is such that $\pi_0 \equiv d\tau_0$. The local inhomogeneous coordinates on $\mathbb{R}P^n$ are the coordinates $\kappa^k = \tau_k/\tau_0$ $(k = 1, \ldots, n)$. Then, at each point $p \equiv (\kappa^1, \ldots, \kappa^n) \in \mathbb{R}P^n$, we define the tangent projective space as a sub-space of a vector space $\mathbb{R}^{n+1}$ with the canonical vector basis $\{\xi_0, \xi_1, \ldots, \xi_n\}$ where

$$\xi_0 \equiv \partial_0 \equiv \frac{\partial}{\partial \tau_0}\,, \qquad \xi_i \equiv \partial_i \equiv \frac{\partial}{\partial \tau_i} \quad (i = 1, \ldots, n)\,. \tag{3.51}$$

Considering ξ_0 as an affine point in $\mathbb{R}^{n+1}$, then we have that $\xi_0 \in P_0$ because ξ_0 has the homogeneous coordinate τ_0 such that $\tau_0 = 1$. Then, we define ξ such that

$$\xi = \xi_0 + \sum_{i=1}^{n} \kappa^i \xi_i\,. \tag{3.52}$$

With this definition of ξ, we identify the origin ξ of $\mathfrak{P}_0(p) \simeq P_0$ with the point $p \in P_0$, and then, the vector field ξ is an embedding in $T\mathbb{R}^{n+1}$. A projective frame $\mathfrak{F}_p$ at each point p can be defined as $\mathfrak{F}_p \equiv \{[\xi], [\xi_1], \ldots, [\xi_n], [\xi_{n+1}]\}$.[30] Obviously, we have $d\kappa^j(\xi) \equiv 1/\tau_0^2(\tau_0\, d\tau_j - \tau_j\, d\tau_0)(\xi) = 0$, and therefore, the cobasis $\mathfrak{B}^* \equiv \{\pi, d\kappa^1, \ldots, d\kappa^n\}$ is the dual of the basis $\mathfrak{B} = \{\xi, \tau_0\, \xi_1, \ldots, \tau_0\, \xi_n\}$. This frame is associated with a projective frame defined with $\xi_{n+1} \equiv \xi + \sum_{i=1}^{n} \tau_0\, \xi_i$. This formula is used mainly when we make a change of projective frame only. Then, the dual cobasis is $\{\pi, d\kappa^1, \ldots, d\kappa^n\}$ where the Yano-Ishihara form (*i.e.*, the projecting form) π is very simply defined as the exact 1-form π_0, *i.e.*, $\pi \equiv d\tau_0$; the latter being constant over $\mathbb{R}P^n$ and vertical by definition.

In this paragraph, the latin indices still represent indices from 1 to n whereas the greek

[30] We use the notation $\mathfrak{F}$, $\mathfrak{B}$ and $\mathfrak{v}$ for frames, bases and vectors, but more correctly we should use rather the notation $\mathcal{F}$, $\mathcal{B}$ and v since there are frames, bases and vectors relative to charts and coordinates (see Remark 6, p.52).

indices are utilized for indices from 0 to n. Then, we have also

$$\mathfrak{v}_0 \equiv \xi\,, \tag{3.53a}$$

$$\mathfrak{v}_i \equiv \tau_0\,\xi_i \equiv \frac{\partial}{\partial\kappa^i}\,, \tag{3.53b}$$

$$\mathfrak{v}_{n+1} \equiv \sum_{\alpha=0}^{n} \mathfrak{v}_\alpha\,. \tag{3.53c}$$

Hence, let $\not\omega_{\mathfrak{B}} \equiv (\not\omega_\alpha^\beta)$ be any Euclidean horizontal connection 1-form over $\mathbb{R}P^n$ and thus, depending only linearly on the 1-forms $d\kappa^j$. Then, in the basis $\{\xi, \tau_0\,\xi_1, \dots, \tau_0\,\xi_n\}$ we have the fundamental relation:

$$\pi(\not\nabla\xi) = \not\omega_0^0\,, \tag{3.54}$$

where $\not\nabla_{\mathfrak{u}}\mathfrak{v}_\alpha \equiv \not\omega_\alpha^\beta(\mathfrak{u})\,\mathfrak{v}_\beta$ for any $\mathfrak{u} \in \chi(\mathbb{R}P^n)$. Therefore, from the relation *1* in theorem 5, we deduce that the projective connection $\omega_{\mathfrak{B}} \equiv (\omega_\alpha^\beta)$ in the basis $\mathfrak{B} = \{\xi, \tau_0\,\xi_1, \dots, \tau_0\,\xi_n\}$ is defined from $\not\omega$ by the relation:

$$\omega \equiv \not\omega - \not\omega_0^0\,\mathbb{1}_{n+1}\,. \tag{3.55}$$

Hence, we have in particular $\omega_0^0 = 0$. Moreover, we define ω such that $\nabla_{\mathfrak{u}}\xi = \mathfrak{q}_0(\mathfrak{u})$ where $\mathfrak{q}_0 \equiv \sum_{i=1}^n \mathfrak{v}_i \otimes \mathfrak{v}^{*,i}$ and $\mathfrak{v}^{*,i} \equiv d\kappa^i$. Therefore, $\nabla_{\mathfrak{u}}\xi = \omega_0^\alpha(\mathfrak{u})\,\mathfrak{v}_\alpha = \omega_0^i(\mathfrak{u})\,\mathfrak{v}_i = \mathfrak{q}_0(\mathfrak{u}) = \underline{\mathfrak{u}} = (i_{\mathfrak{u}}d\kappa^j)\,\mathfrak{v}_j$. It follows that

$$\omega_0^i \equiv d\kappa^i\,. \tag{3.56}$$

Moreover, we can take ω such that $Tr\,\omega_{\mathfrak{B}} = 0$, *i.e.*, $\sum_{i=1}^n \omega_i^i = \sum_{i=1}^n(\not\omega_i^i - \not\omega_0^0) = 0$. This condition can be explained in details by exhibiting the constraint (3.39) or equivalently the relation *1* in theorem 5. Indeed, the section $\mathfrak{C}$ is a change of projective frame but leaving $[\xi] \equiv [\mathfrak{v}_0]$ invariant, and, moreover, with $\det\mathfrak{C} \neq 0$. Thus, let $\mathfrak{v}'_\alpha$ be the new vectors such that:

$$\mathfrak{v}'_0 = \mathfrak{C}(\mathfrak{v}_0) = \mathfrak{v}_0\,, \qquad \mathfrak{v}'_i = \mathfrak{C}(\mathfrak{v}_i) = \mathfrak{v}_i + \mathfrak{C}_i\,\mathfrak{v}_0\,, \tag{3.57}$$

where the $\mathfrak{C}_k$'s are real functions of p. Then, from this change of basis, we deduce that

$$\not\nabla\mathfrak{v}'_0 = \mathfrak{v}'_0 \otimes \left(\not\omega_0^0 - \mathfrak{C}_k\,\not\omega_0^k\right) + \mathfrak{v}'_k \otimes \not\omega_0^k\,, \tag{3.58a}$$

$$\not\nabla\mathfrak{v}'_i = \mathfrak{v}'_0 \otimes \left(d\mathfrak{C}_i + \mathfrak{C}_i\,\not\omega_0^0 + \not\omega_i^0 - \mathfrak{C}_k\,(\not\omega_i^k + \mathfrak{C}_i\,\not\omega_0^k)\right) + \mathfrak{v}'_k \otimes (\not\omega_i^k + \mathfrak{C}_i\,\not\omega_0^k)\,. \tag{3.58b}$$

And therefore, the diagonal coefficients $\omega^i_i \equiv \not\omega^i_i - \not\omega^0_0$ of the projective connection are changed to

$$\omega'^i_i \equiv \not\omega'^i_i - \not\omega'^0_0 = \not\omega^i_i - \not\omega^0_0 + \mathfrak{C}_i \not\omega^i_0 + \sum_{k=1}^n \mathfrak{C}_k \not\omega^k_0 \tag{3.59}$$

whatever are the $\mathfrak{C}_k$'s. Then, to obtain a traceless projective connection ω', we must find functions $\mathfrak{C}_k$ such that

$$\sum_{i=1}^n \omega'^i_i = \sum_{i=1}^n (\not\omega^i_i - \not\omega^0_0) + (n+1) \sum_{k=1}^n \mathfrak{C}_k \not\omega^k_0 = 0\,, \tag{3.60}$$

where the coefficients $\not\omega^\beta_\alpha$ are $\mathcal{O}_{\mathbb{R}P}$-linear combinations of the n horizontal 1-forms $d\kappa^i$ $(i = 1, \ldots, n)$.

We can always find a set of n functions $\mathfrak{C}_k$ because there is a system (3.60) of n algebraic equations only to satisfy. Indeed, the connection coefficients are 1-forms that are linear combinations of the fundamental forms $d\kappa^i$, and therefore, the functions $\mathfrak{C}_k$ can be found univocally and the new projective connections ω' becomes traceless as expected in the new projective frame $\{[\mathfrak{v}'_0], \ldots, [\mathfrak{v}'_n]\}$.

Moreover, taking a traceless projective connection ω on the basis $\{\mathfrak{v}_0, \ldots, \mathfrak{v}_n\}$ with $\omega^0_0 = 0$, and then, applying another change of projective frame $\widetilde{\mathfrak{C}}$ such that

$$\mathfrak{v}'_0 = \widetilde{\mathfrak{C}}(\mathfrak{v}_0) = \mathfrak{v}_0 + \sum_{k=1}^n \widetilde{\mathfrak{C}}^k_0\, \mathfrak{v}_k\,, \qquad \mathfrak{v}'_i = \widetilde{\mathfrak{C}}(\mathfrak{v}_i) = \sum_{j=1}^n \widetilde{\mathfrak{C}}^j_i\, \mathfrak{v}_j\,, \tag{3.61}$$

where $\det(\widetilde{\mathfrak{C}}^j_i) \neq 0$ (which involves the condition for embedding; see Lemma 2, p.60), we can easily find $\widetilde{\mathfrak{C}}$ such that

$$\mathfrak{q}_0 \equiv \sum_{k=1}^n \mathfrak{v}_k \otimes \not\omega^k_0 = \sum_{k=1}^n \mathfrak{v}'_k \otimes d\kappa^k\,, \tag{3.62}$$

and thus, such that the new projective connection satisfies the relations (3.56). In conclusion, in the basis $\mathfrak{B} = \{\xi, \tau_0\,\xi_1, \ldots, \tau_0\,\xi_n\}$, the more general $\mathfrak{B}$-complete projective Cartan connection ω obtained from any Euclidean connection $\not\omega$ such that $Tr\,\not\omega_{\mathfrak{B}} = (n+1)\not\omega^0_0$ is of the following form:

$$\omega_{\mathfrak{B}} \equiv \begin{pmatrix} 0 & d\kappa^1 & d\kappa^2 & \vdots & d\kappa^n \\ \not\omega_1^0 & \not\omega_1^1 - \not\omega_0^0 & \not\omega_1^2 & \vdots & \not\omega_1^n \\ \not\omega_2^0 & \not\omega_2^1 & \not\omega_2^2 - \not\omega_0^0 & \vdots & \not\omega_2^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \not\omega_n^0 & \not\omega_n^1 & \not\omega_n^2 & \vdots & \not\omega_n^n - \not\omega_0^0 \end{pmatrix}, \qquad Tr\,\omega_{\mathfrak{B}} = 0\,, \qquad i_\xi \omega = 0\,, \tag{3.63}$$

where $d\kappa^i(\xi_j) \equiv \delta^i_j/\tau_0$.

Remark 11. *Also, we mention the important well-known discrepancy between matrices multiplication in the space of 1-forms and in the space of vectors. Indeed, in the sequel, we have changes of coordinates between 1-forms or between vectors and involving matrices. According to our notations for indices of matrices indicated above, we can have, for instance, the following relations:*

$$dz^\alpha = \sum_\beta A^\alpha_\beta\, dy^\beta\,, \qquad dy^\beta = \sum_\mu B^\beta_\mu\, dx^\mu\,, \tag{3.64}$$

and by duality, we obtain also that

$$\frac{\partial}{\partial z^\mu} = \sum_\nu A^{-1,\nu}_\mu \frac{\partial}{\partial y^\nu}\,, \qquad \frac{\partial}{\partial y^\rho} = \sum_\sigma B^{-1,\sigma}_\rho \frac{\partial}{\partial x^\sigma}\,. \tag{3.65}$$

Then, we translate these previous expressions using the following convention of notation to differentiate the right and left multiplication of matrices:

$${}^t dz = {}^t dy\,.\,A\,, \qquad {}^t dy = {}^t dx\,.\,B\,, \qquad \frac{\partial}{\partial z} = A^{-1}\cdot\frac{\partial}{\partial y}\,, \qquad \frac{\partial}{\partial y} = B^{-1}\cdot\frac{\partial}{\partial x}\,, \tag{3.66}$$

where ${}^t dx$ *is the transpose of* dx*. Indeed, if we use the left multiplication of matrices for vectors, i.e., we have that:*

$$\frac{\partial}{\partial z} = A^{-1}\,.\,B^{-1}\cdot\frac{\partial}{\partial x} = (B\,A)^{-1}\cdot\frac{\partial}{\partial x}\,, \tag{3.67}$$

but we must have the right multiplication of matrices for the corresponding 1-forms, since we have that

$${}^t dz = {}^t dy\,.\,A = {}^t dx\,.\,B\,.\,A \neq {}^t dx\,.\,A\,.\,B\,. \tag{3.68}$$

Indeed, if ${}^t dz = {}^t dx \,.\, A \,.\, B$ *then, from* (3.67)*, we would have the following inconsistency and with formula* (27)*:*

$$dz^\alpha \left(\frac{\partial}{\partial z^\rho}\right) = \sum_{\beta,\tau} (A\,.\,B)^\alpha_\beta \,(B\,.\,A)^{-1,\tau}_\rho \, dx^\beta \left(\frac{\partial}{\partial x^\tau}\right) = \sum_{\beta,\tau,\mu,\sigma} A^\alpha_\mu \, B^\mu_\beta \, A^{-1,\tau}_\sigma \, B^{-1,\sigma}_\rho \, \delta^\beta_\tau \neq \delta^\alpha_\rho \,. \quad (3.69)$$

IV. THE THREE-DIMENSIONAL SUBMANIFOLDS OF $\mathcal{M}_{RPS}$ MODELED ON $\mathbb{R}P^3$

A. The $\mathbb{R}P^3$ projective connections on $\mathcal{M}_{RPS}$

In this paragraph, the latin indices represent indices from 2 to 4 whereas the greek indices are utilized for indices from 1 to 4.

Also, we consider that $M \equiv \mathcal{M}_{RPS}$ is a four dimensional manifold experienced by users of a RPS (Relativistic Positioning System). It does not correspond to the set of all events in spacetime because RPSs forbid the localization of any kind of spacetime event but only those solely *experienced* by users where they are and not those they observe afar and/or localize. Thus, although four times stamps, i.e., four coordinates are accounting in this geometry, it cannot be the “true” geometry of the spacetime. For this, we will need to add another fifth time stamp meaning that the “true” spacetime can only be geometrized as a four dimensional manifold *embedded* in a five dimensional manifold.

In this section, we consider that $\mathbb{R}P^3$ is the model space soldered to three dimensional submanifolds $\mathcal{J}$ foliating $\mathcal{M}_{RPS}$ and the question rises to understand the meaning of a local (tangent) soldered three-dimensional projective structure. Actually, this can be related to the equivalence principle within the context of the local space and time splitting. Indeed, any local physical observation of events in spacetime is implicitly based to such local (model) space attached to any observer. We think that soldering a model space is an expression of the equivalence principle and, moreover, that the space and time splitting is an expression of the projective structure. This is particularly true considering the interpretations of the 4-velocities which must linked to space-like 3-velocities; the former being the homogeneous vectors of which the latter are the inhomogeneous projective parts. Hence, the three dimensional projective structure is essentially due to physical observations always brought back, somehow, to non-relativistic

Galilean frames. The main result we obtain in the next section is that the three-dimensional space-like submanifolds cannot be conformally flat (see also Appendix F, p.217, for a deeper insight into this projective feature).

Then, our goal is to symmetrize any expression because the coordinates τ_j must be on the same status, *i.e.*, we must not discriminate one of them. This differs from the previous paragraph with the Cartan exemple. We start with the Euclidean vector space $\mathbb{R}^4$ with the local homogeneous time coordinates τ_α.

Besides, symmetrization by the Galois group S_4 appears in the expressions for the coefficients of any univariate polynomial equation of degree four with four different roots τ_α. The normal Galois subgroup S_3 is the coset of $S_4/(\mathbb{Z}_2\times\mathbb{Z}_2)$ and is the Galois group of the *cubic Lagrange resolvent* associated with quartic equations. It is the symmetry group for the expressions of the coefficients of this cubic equation if its roots are different again. Its three roots γ_i can be expressed from the roots τ_α of the initial quartic equation with the following expressions (well-known when solving quartic equations):

$$\begin{aligned}
\gamma_1 &= \tau_1\tau_2 + \tau_3\tau_4\,,\\
\gamma_2 &= \tau_1\tau_3 + \tau_2\tau_4\,,\\
\gamma_3 &= \tau_1\tau_4 + \tau_3\tau_2\,.
\end{aligned} \tag{4.1}$$

Then, we define the coordinates ρ_i such that $\rho_{i+1} \equiv 2\,\gamma_i$ $(i = 1, 2, 3)$. This defines a local diffeomorphism $K : \tau \to \rho$ between the coordinates τ_α and four other coordinate ρ_β such that

$$\rho = K(\tau) \equiv \hat{\tau}\,.\,\tau \tag{4.2}$$

where

$$\rho \equiv \begin{pmatrix}\rho_1\\ \rho_2\\ \rho_3\\ \rho_4\end{pmatrix}, \quad \tau \equiv \begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_4\end{pmatrix}, \quad \hat{\tau} \equiv \begin{pmatrix}\tau_1 & \tau_2 & \tau_3 & \tau_4\\ \tau_2 & \tau_1 & \tau_4 & \tau_3\\ \tau_3 & \tau_4 & \tau_1 & \tau_2\\ \tau_4 & \tau_3 & \tau_2 & \tau_1\end{pmatrix} = \sum_{\alpha=1}^{4} \tau_\alpha\,\hat{h}^\alpha. \tag{4.3}$$

The local diffeomorphism K is locally invertible (see Appendix I).[31] The set of matrices $\hat{u}$ such that $\hat{u} \equiv \sum_{i=2}^{4} u_i\,\hat{h}^i$ for $u_i \in \mathbb{R}$ forms the simplest finite real special Jordan algebra $\mathfrak{J}_2$ generated

[31] Another interesting possibility: $\hat{\rho} \equiv \ln\hat{\tau}$. With such definition, we obtain $d\hat{\tau} \equiv d\hat{\rho}\,.\,\hat{\tau}$ and $\partial_\rho \equiv \hat{\tau}\,.\,\partial_\tau$.

by two elements and of order 3. We have the following relations ($i \neq j \neq k$; $i, j, k = 2, 3, 4$):

$$\hat{h}^i \hat{h}^j = \hat{h}^j \hat{h}^i = \hat{h}^k, \qquad (\hat{h}^i)^2 = \hat{h}^1 = \mathbb{1}, \tag{4.4}$$

Therefore, we obtain also the Klein group $V = \mathbb{Z}_2 \times \mathbb{Z}_2$ with the elements $\{\mathbb{1}, \hat{h}^2, \hat{h}^3, \hat{h}^4\}$. Then, two matrices $\hat{\tau}$ and $\hat{\chi}$ commute, *i.e.*, we have

$$\hat{\tau}\,\hat{\chi} = \hat{\chi}\,\hat{\tau}\,, \tag{4.5}$$

and thus, the associative algebra $\mathfrak{J}_2$ is also commutative. And then, we deduce the following trace formulas and the metric η such that ($\alpha, \beta = 1, \ldots, 4$):

$$\frac{1}{4}\,Tr(\hat{h}^\alpha\,\hat{h}^\beta) = \delta^{\alpha\beta}\,, \tag{4.6a}$$

$$\frac{1}{4}\,Tr(\hat{h}^\alpha\,\hat{\tau}) = \tau_\alpha\,, \tag{4.6b}$$

$$\frac{1}{4}\,Tr(\hat{\tau}\,\hat{\chi}\,\hat{\mu}) = {}^t\tau\,\hat{\mu}\,\chi\,, \tag{4.6c}$$

$$\eta(\tau, \chi) = \frac{1}{4}\,Tr(\hat{\tau}\,\hat{\chi}\,\widehat{\mathfrak{H}}\,) = \sum_{\alpha\neq\beta=1}^{4} \tau_\alpha\,\chi_\beta = {}^t\tau\,\widehat{\mathfrak{H}}\,\chi\,, \tag{4.6d}$$

where $\widehat{\mathfrak{H}} = \hat{h}^2 + \hat{h}^3 + \hat{h}^4$ is the following matrix:

$$\widehat{\mathfrak{H}} \equiv \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{4.7}$$

Moreover, it matters to notice the following important property:

$$\hat{\tau}^{-1}\,\widehat{\mathfrak{H}}\,\hat{\tau} = \widehat{\mathfrak{H}}\,. \tag{4.8}$$

We can define [Jac63] a generic norm $N(\tau)$ on $\mathcal{M}_{RPS}$ such that

$$N(\tau) = \Big|(\tau_1 + \tau_2 + \tau_3 + \tau_4)(\tau_1 - \tau_2 - \tau_3 + \tau_4)(\tau_1 + \tau_2 - \tau_3 - \tau_4)(\tau_1 - \tau_2 + \tau_3 - \tau_4)\Big|\,. \tag{4.9}$$

We see that $N(\tau) = |\det\hat{\tau}|$ and we have the fundamental result:

$$N(\rho) = N(\tau)^2\,. \tag{4.10}$$

The set of elements $\hat{\tau}$ constitute the normed unital algebra $\mathfrak{U}_2$ associated with the Jordan algebra $\mathfrak{J}_2$, and we write indifferently the norm of τ as $N(\tau)$ or $N(\hat{\tau}) \equiv N(\tau)$ and we extend naturally the product of matrices $\hat{\tau}$ on $\mathfrak{U}_2$. Then, we obtain in particular $N(\hat{\tau}\,\hat{\mu}) = N(\hat{\tau})\,N(\hat{\mu})$. Moreover, we define the inverse τ^{-1} of τ such that $\widehat{\tau^{-1}} = \hat{\tau}^{-1}$, and then, $\hat{\tau}^{-1}.\,\hat{\tau} = \hat{\tau}\,.\,\hat{\tau}^{-1} = \mathbb{1}$. Thus, we obtain

$$\tau^{-1} \equiv \frac{1}{N(\tau)} \begin{pmatrix} (\tau_1)^3 - \tau_1(\tau_2)^2 - \tau_1(\tau_3)^2 - \tau_1(\tau_4)^2 + 2\tau_2\tau_3\tau_4 \\ (\tau_2)^3 - \tau_2(\tau_3)^2 - \tau_2(\tau_1)^2 - \tau_2(\tau_4)^2 + 2\tau_3\tau_4\tau_1 \\ (\tau_3)^3 - \tau_3(\tau_4)^2 - \tau_3(\tau_2)^2 - \tau_3(\tau_1)^2 + 2\tau_4\tau_1\tau_2 \\ (\tau_4)^3 - \tau_4(\tau_1)^2 - \tau_4(\tau_3)^2 - \tau_4(\tau_2)^2 + 2\tau_1\tau_2\tau_3 \end{pmatrix}. \tag{4.11}$$

From the multiplication on $\mathfrak{U}_2$, we see also that

$$\hat{\rho} = \hat{\tau}\,.\,\hat{\tau}\,. \tag{4.12}$$

In addition, the matrix $\hat{\tau}$ is diagonalizable in the matrix $d_\tau = P^{-1}\,\hat{\tau}\,P$ such that

$$d_\tau \equiv \begin{pmatrix} \tau_1 - \tau_2 + \tau_3 - \tau_4 & 0 & 0 & 0 \\ 0 & \tau_1 - \tau_2 - \tau_3 + \tau_4 & 0 & 0 \\ 0 & 0 & \tau_1 + \tau_2 - \tau_3 - \tau_4 & 0 \\ 0 & 0 & 0 & \tau_1 + \tau_2 + \tau_3 + \tau_4 \end{pmatrix}, \tag{4.13}$$

and

$$P \equiv \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \end{pmatrix}. \tag{4.14}$$

Besides, in the Euclidean space $\mathbb{R}^3$ with the Cartesian coordinates (x, y, z), the surface defined by the equation $x\,y\,z = 1$ is known as the so-called *Tzitzeica surface* (see Figure IV.1) with four strata.

The relation $N(\hat{\tau}) = 1$ defines a three-dimensional variety homeomorphic, up to a linear change of coordinates, to what we call a '*generalized Tzitzeica surface*' and denoted by $\mathcal{T}i^3$. Hence, K defines a covering $\mathcal{T}i^3 \longrightarrow \mathcal{T}i^3$ with eight inverse maps corresponding to the eight strata of $\mathcal{T}i^3$ (see Appendix I, p.230).

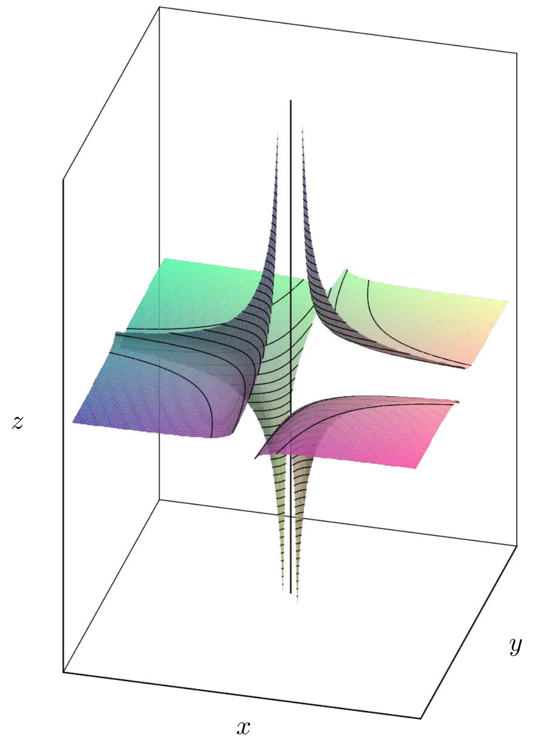


Figure IV.1. The Tzitzeica surface $\mathcal{T}\imath^2$ in $\mathbb{R}^3$ with its four strata.

Then, we consider the ρ_α's to be the homogeneous coordinates while the κ^i's will be the inhomogeneous ones. The representing affine space P_0 is the "eighth" $S^3/(\mathbb{Z}_2)^3$ of a sphere S^3 defined by $\rho_1 = \sum_{\alpha=1}^4 (\tau_\alpha)^2 = 1$ with $\tau_\beta > 0$ ($\beta = 1, \ldots, 4$), and a projecting form $\tilde{\pi}_1$ such that $\tilde{\pi}_1 \equiv d\rho_1 = 2\sum_{\alpha=1}^4 \tau_\alpha\, d\tau_\alpha$. The local inhomogeneous coordinates on $\mathbb{R}P^3$ are the coordinates

$$\kappa^k \equiv \rho_k/\rho_1 \qquad (k = 2, 3, 4) \tag{4.15}$$

of the point $p \in \mathbb{R}P^3$. In $\mathbb{R}^4$, we take the canonical basis $\{\xi_1, \xi_2, \xi_3, \xi_4\}$ such that $d\rho_\alpha(\xi_\beta) = \delta_{\alpha,\beta}$ where ${}^t d\rho = 2\, {}^t d\tau \,.\, \hat{\tau}$ or, equivalently,

$$d\hat{\rho} = 2\,\hat{\tau}.\, d\hat{\tau}\,. \tag{4.16}$$

We denote also the dual basis by $\{\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \tilde{\pi}_4\}$ where $\tilde{\pi}_\alpha \equiv d\rho_\alpha = 2\,({}^t d\tau \,.\, \hat{\tau})_\alpha = 2\sum_{\beta=1}^4 \hat{\tau}^\alpha_\beta\, d\tau_\beta$. Thus, we obtain that

$$\tilde{\xi}_\beta \equiv \partial_{\rho_\beta} \equiv \frac{\partial}{\partial \rho_\beta} = \frac{1}{2}\,(\hat{\tau}^{-1} \,.\, \partial_\tau)_\beta = \frac{1}{2}\sum_{\alpha=1}^4 \hat{\tau}^{-1,\alpha}_\beta\, \partial_{\tau_\alpha}\,, \tag{4.17}$$

where ∂_τ is a vector of which the components are the partial derivatives $\partial_{\tau_\beta} \equiv \frac{\partial}{\partial \tau_\beta}$ with respect to the τ_β's. The vectors ξ_α are not defined for $N(\hat{\tau}) = 0$ because they are defined with the inverse of $\hat{\tau}$. For instance, if $\tau_1 + \tau_2 = \tau_3 + \tau_4$, then, $\rho_1 + \rho_2 = \rho_3 + \rho_4$ and $N(\rho) = N(\tau) = 0$, and then, we have a "descent" from $\mathbb{R}P^3$ to $\mathbb{R}P^2$. The Yano-Ishihara (projecting) form π and ξ are such that

$$\pi \equiv \pi_1 = \tilde{\pi}_1/\rho_1 = d\kappa^1 \,, \tag{4.18}$$

where

$$\kappa^1 = \ln \rho_1 \,, \tag{4.19}$$

and

$$\xi \equiv \xi_1 = e^{\kappa^1}\, \tilde{\xi} \,, \tag{4.20}$$

where

$$\tilde{\xi} = \tilde{\xi}_1 + \sum_{i=2}^{4} \kappa^i\, \tilde{\xi}_i \,. \tag{4.21}$$

The projective frame $\mathfrak{F}_p$ of the tangent vector space $T_p\mathbb{R}^4$ at each point $p \in P_0$ is obviously defined as $\mathfrak{F}_p = \{[\tilde{\xi}], [\tilde{\xi}_2], [\tilde{\xi}_3], [\tilde{\xi}_4], [\tilde{\xi}_5]\}$ with $\tilde{\xi}_5 \equiv \tilde{\xi} + \sum_{i=2}^{4} \tilde{\xi}_i$. We have $d\kappa^j(\xi) = 0$ for $j = 2, 3, 4$, and therefore, the cobasis $\mathfrak{B}^* \equiv \{\pi, d\kappa^2, d\kappa^3, d\kappa^4\}$ is the dual of the basis $\mathfrak{B} = \{\xi, e^{\kappa^1}\, \tilde{\xi}_2, e^{\kappa^1}\, \tilde{\xi}_3, e^{\kappa^1}\, \tilde{\xi}_4\}$ for $T_p\mathbb{R}^4$ (see precedent paragraph).

Then, from the Cartan approach above, if $\not\omega(\rho)$ is an Euclidean connection form on $\mathbb{R}^4$ such that $Tr(\not\omega_{\mathfrak{B}}) = 4\not\omega_1^1$, then, the projective Cartan connection ω on $\mathbb{R}P^3$ is such that in the basis $\mathfrak{B}$ we obtain

$$\omega_{\mathfrak{B}} \equiv \begin{pmatrix} 0 & d\kappa^2 & d\kappa^3 & d\kappa^4 \\ \not\omega_2^1 & \not\omega_2^2 - \not\omega_1^1 & \not\omega_2^3 & \not\omega_2^4 \\ \not\omega_3^1 & \not\omega_3^2 & \not\omega_3^3 - \not\omega_1^1 & \not\omega_3^4 \\ \not\omega_4^1 & \not\omega_4^2 & \not\omega_4^3 & \not\omega_4^4 - \not\omega_1^1 \end{pmatrix} , \qquad Tr(\omega_{\mathfrak{B}}) = 0 \,, \qquad i_\xi \omega = 0 \,. \tag{4.22}$$

In this expression for ω, the 1-forms $\not\omega_\alpha^\beta$ depend only on the variables κ^k and the 1-forms $d\kappa^h$. Also, the diagonal components are differences between diagonal components of $\not\omega_{\mathfrak{B}}$.

Let $f \equiv (f^\alpha_\beta)$ be the matrix for the change of basis from $\mathfrak{B} = \{\mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4\} = \{\xi, e^{\kappa^1}\,\tilde{\xi}_2, e^{\kappa^1}\,\tilde{\xi}_3, e^{\kappa^1}\,\tilde{\xi}_4\}$ to $\mathfrak{B}' \equiv \{\mathfrak{v}'_1, \mathfrak{v}'_2, \mathfrak{v}'_3, \mathfrak{v}'_4\} = \{\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4\}$, then, we have:

$$f \equiv \frac{1}{\rho_1}\begin{pmatrix} 1 & -\kappa^2 & -\kappa^3 & -\kappa^4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad f^{-1} \equiv \rho_1 \begin{pmatrix} 1 & \kappa^2 & \kappa^3 & \kappa^4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.23}$$

where $\mathfrak{v}'_\alpha = \sum_{\beta=1}^4 f^\beta_\alpha\,\mathfrak{v}_\beta$, and where f^β_α is by convention the coefficient on the raw α and the column β.

Then, let E be an endomorphism on $\mathbb{R}^4$, then, E can be represented by the matrix $E_{\mathfrak{B}} \equiv (E^\beta_\alpha)$ in the basis $\mathfrak{B}$ where $E^\beta_\alpha = \mathfrak{v}^{*,\beta}(E(\mathfrak{v}_\alpha))$ is the component of the matrix $E_{\mathfrak{B}}$ at the column β and the raw α. In the basis $\mathfrak{B}' \equiv R_f\mathfrak{B}$, E is represented by the matrix $E_{\mathfrak{B}'} \equiv (E'^\beta_\alpha)$ where $E'^\beta_\alpha = \mathfrak{v}'^{*,\beta}(E(\mathfrak{v}'_\alpha))$. Therefore, we have $E'^\beta_\alpha = \sum_{\mu,\nu=1}^4 (f^{-1})^\beta_\mu\, f^\nu_\alpha\, E^\mu_\nu$, *i.e.*, $E_{\mathfrak{B}'} = f^{-1} E_{\mathfrak{B}}\, f$. Also, from $\mathfrak{v}^{*,\rho} = \sum_{\beta=1}^4 f^\rho_\mu\,\mathfrak{v}'^{*,\mu}$, we deduce for any vector $\mathfrak{u}$ that $\mathfrak{u}^\alpha_{\mathfrak{B}'} = \sum_{\beta=1}^4 (f^{-1})^\alpha_\beta\,\mathfrak{u}^\beta_{\mathfrak{B}}$, and then, if we represent $\mathfrak{u}_{\mathfrak{B}}$ by a one column matrix

$$\mathfrak{u}_{\mathfrak{B}} = \begin{pmatrix} \mathfrak{u}^1_{\mathfrak{B}} \\ \vdots \\ \mathfrak{u}^4_{\mathfrak{B}} \end{pmatrix}, \tag{4.24}$$

then, we have $\mathfrak{u}_{\mathfrak{B}'} = f^{-1} \,.\, \mathfrak{u}_{\mathfrak{B}}$. And for a co-vector $\mathfrak{u}^*$, we obtain $\mathfrak{u}^*_{\mathfrak{B}'} = \mathfrak{u}^*_{\mathfrak{B}} \,.\, f$ where $\mathfrak{u}^*_{\mathfrak{B}}$ is represented by a one raw matrix

$$\mathfrak{u}^*_{\mathfrak{B}} = (\mathfrak{u}^*_{\mathfrak{B},1}, \ldots, \mathfrak{u}^*_{\mathfrak{B},4}) \,. \tag{4.25}$$

In particular, in the basis $\mathfrak{B}$, the basis vector $\mathfrak{v}_i$ are represented by the the canonical vectors

$$\mathfrak{v}_{\mathfrak{B},i} = \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \tag{4.26}$$

of the Euclidean space, where the number 1 is at the i-th position from the top. This is the formula we will use for the changes of representation. In addition, we note that we have not any relation of the form "$(R_f\omega)_{\mathfrak{B}} = \omega_{\mathfrak{B}'} = f^{-1}\omega_{\mathfrak{B}}\, f = Ad(f^{-1})\,\omega_{\mathfrak{B}}$" for a connection ω because in the present case we have a change of coordinates rather than a change of basis (see details in Remark 6, p.52).

Also, we can notice that f is not a change of projective frame because $\rho_1\tilde{\xi}_5 \equiv \rho_1(\tilde{\xi}+\sum_{i=2}^4 \tilde{\xi}_i)$ is not collinear to $\xi'_5 \equiv \sum_{\alpha=1}^4 \tilde{\xi}_\alpha$.

Beginning with the projectively equivalent projective connection

$$\overset{\circ}{\omega} \equiv \omega + \mathfrak{r}\,\mathbb{1} \tag{4.27}$$

(with $\mathfrak{r}$ a horizontal 1-form defined over $\mathbb{R}P^3$ and thus such that $i_\xi\mathfrak{r} = 0$) associated with the projective covariant derivative ∇, we obtain the connection $\overset{\circ}{\omega}$ in the basis $\mathfrak{B}' \equiv \{\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4\}$ applying the formula (3.34) (p.53) associated with the transition morphism $t_{2,1} \equiv f$. Then, we deduce that

$$\overset{\circ}{\omega}_{\mathfrak{B}'} = f^{-1}.\, df + f^{-1}\omega_{\mathfrak{B}}\, f + \mathfrak{r}\,\mathbb{1}\,. \tag{4.28}$$

Also, let us notice that, in this very particular case with f only, we have $df\,.\,f^{-1} = f^{-1}\,.\,df$. And, in the basis $\{\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3, \tilde{\xi}_4\}$ we have that $\overset{\circ}{\omega}$ is of the following form:

$$\overset{\circ}{\omega}_{\mathfrak{B}'} \equiv \widehat{\omega}_{\kappa,\mathfrak{B}'} + (\mathfrak{r} - d\kappa^1)\,\mathbb{1}\,, \tag{4.29}$$

with $Tr(\overset{\circ}{\omega}_{\mathfrak{B}'}) = 4\,(\mathfrak{r}-d\kappa^1)$ since $Tr(\omega_{\mathfrak{B}}) = 0$, $Tr(f^{-1}\,.\,df) = -4d\kappa^1$, $\kappa^1 \equiv \ln\rho_1$ and where $\widehat{\omega}_{\kappa,\mathfrak{B}'}$, depending only on κ^i $(i = 1, 2, 3)$, *i.e.*, inhomogeneous, is such that $Tr\,\widehat{\omega}_{\kappa,\mathfrak{B}'} = 0$, $i_\xi\widehat{\omega}_{\kappa,\mathfrak{B}'} = 0$ and

$$\widehat{\omega}_{\kappa,\mathfrak{B}'} \equiv \begin{pmatrix} \sum_{i=2}^4 \omega_i^1\kappa^i & \sum_{k=2}^4 \omega_k^2\kappa^k - \kappa^2\big(\omega_1^1 + \sum_{i=2}^4 \omega_i^1\kappa^i\big) & \sum_{k=2}^4 \omega_k^3\kappa^k - \kappa^3\big(\omega_1^1 + \sum_{i=2}^4 \omega_i^1\kappa^i\big) & \sum_{k=2}^4 \omega_k^4\kappa^k - \kappa^4\big(\omega_1^1 + \sum_{i=2}^4 \omega_i^1\kappa^i\big) \\ \omega_2^1 & \omega_2^2 - \omega_1^1 - \omega_2^1\kappa^2 & \omega_2^3 - \omega_2^1\kappa^3 & \omega_2^4 - \omega_2^1\kappa^4 \\ \omega_3^1 & \omega_3^2 - \omega_3^1\kappa^2 & \omega_3^3 - \omega_1^1 - \omega_3^1\kappa^3 & \omega_3^4 - \omega_3^1\kappa^4 \\ \omega_4^1 & \omega_4^2 - \omega_4^1\kappa^2 & \omega_4^3 - \omega_4^1\kappa^3 & \omega_4^4 - \omega_1^1 - \omega_4^1\kappa^4 \end{pmatrix}. \tag{4.30}$$

Moreover, all of the horizontal 1-forms coefficients in the matrix (4.30) depend only on the variables κ^k and the 1-forms $d\kappa^h$ $(k,h=2,3,4)$, *i.e.*, there are inhomogeneous.

Now, we apply again the formula (3.34) for the change of basis defined by the relations (4.17) from the basis $\{\mathfrak{v}'_1,\mathfrak{v}'_2,\mathfrak{v}'_3,\mathfrak{v}'_4\}=\{\partial_{\rho_1},\partial_{\rho_2},\partial_{\rho_3},\partial_{\rho_4}\}$ to the basis $\mathfrak{B}''\equiv\{\mathfrak{v}''_1,\mathfrak{v}''_2,\mathfrak{v}''_3,\mathfrak{v}''_4\}\equiv\{\partial_{\tau_1},\partial_{\tau_2},\partial_{\tau_3},\partial_{\tau_4}\}=R_{\hat{\tau}}\mathfrak{B}'$. And thus, we have that $\mathfrak{v}''_\alpha\equiv 2\sum_{\beta=1}^4\hat{\tau}^\beta_\alpha\,\mathfrak{v}'_\beta$, and therefore, the Euclidean connection $\overset{\circ}{\omega}$ in the basis $\mathfrak{B}''\equiv\{\partial_{\tau_1},\partial_{\tau_2},\partial_{\tau_3},\partial_{\tau_4}\}$ is such that

$$\overset{\circ}{\omega}_{\mathfrak{B}''}\equiv\hat{\tau}^{-1}.\,d\hat{\tau}+\hat{\tau}^{-1}\,\overset{\circ}{\omega}_{\mathfrak{B}'}\,\hat{\tau}\,,\tag{4.31}$$

or, equivalently, $\overset{\circ}{\omega}{}^{\alpha}_{\mathfrak{B}'',\beta}\equiv\hat{\tau}^{-1,\alpha}_{\mu}\,d\hat{\tau}^{\mu}_{\beta}+\hat{\tau}^{-1,\alpha}_{\eta}\,\hat{\tau}^{\mu}_{\beta}\,\overset{\circ}{\omega}{}^{\eta}_{\mathfrak{B}',\mu}$ which are tremendous expressions with respect to the $\not{\omega}^\beta_\alpha$'s and the τ_α's. And we obtain in particular:

$$Tr(\hat{\tau}^{-1}.\,d\hat{\tau})=4\sum_{\alpha=1}^{4}(\tau^{-1})_\alpha\,d\tau_\alpha=Tr(P\,d\ln(d_\tau)\,P^{-1})=d\ln(N(\tau))\,,\tag{4.32a}$$

$$Tr(\overset{\circ}{\omega}_{\mathfrak{B}''})=d\ln(N(\tau)\,e^{-4\kappa^1})+4\,\mathfrak{r}\,,\tag{4.32b}$$

where $\rho_1=e^{\kappa^1}=\sum_{\alpha=1}^4\tau_\alpha^2$. Moreover, if ξ is the vector defined in (4.20) again, then we have

$$Tr(\overset{\circ}{\omega}_{\mathfrak{B}'})=4\,(\mathfrak{r}-d\kappa^1)\,,\tag{4.33a}$$

$$i_\xi\overset{\circ}{\omega}_{\mathfrak{B}'}=-\mathbb{1}\,,\tag{4.33b}$$

and

$$\begin{aligned}\hat{\tau}^{-1}.\,i_\xi d\hat{\tau}&=\frac{1}{2}\,\hat{\tau}^{-1}\,\hat{\tau}^{-1}\,(i_\xi d\hat{\rho})=\frac{1}{2}\,\hat{\rho}^{-1}\,(i_\xi d\hat{\rho})\\&=\frac{1}{2}\,\hat{\rho}^{-1}\left(\sum_{\alpha=1}^{4}(i_\xi d\rho_\alpha)\,h^\alpha\right)=\frac{e^{\kappa^1}}{2}\,\hat{\rho}^{-1}\left(\mathbb{1}+\sum_{i=2}^{4}\kappa^i\,h^i\right)\\&=\frac{e^{\kappa^1}}{2}\,\hat{\rho}^{-1}\left(\mathbb{1}+\sum_{i=2}^{4}\frac{\rho_i}{\rho_1}\,h^i\right)=\frac{e^{\kappa^1}}{2}\,e^{-\kappa^1}\;\hat{\rho}^{-1}.\,\hat{\rho}\\&=\frac{1}{2}\,\mathbb{1}\,.\end{aligned}\tag{4.34}$$

Then, in conclusion, we obtain that

$$Tr(\overset{\circ}{\omega}_{\mathfrak{B}''})=d\ln(N(\tau)\,e^{-4\kappa^1})+4\,\mathfrak{r}\,,\tag{4.35a}$$

$$i_\xi\overset{\circ}{\omega}_{\mathfrak{B}''}=-\frac{1}{2}\,\mathbb{1}\,.\tag{4.35b}$$

As a result, $\overset{\circ}{\omega}_{\mathfrak{B}'}$ and $\overset{\circ}{\omega}_{\mathfrak{B}''}$ are not horizontal and not inhomogeneous, and therefore, we will say they are "*conformal*" with respect to $\kappa^1 = \ln\left(\sum_{\alpha=1}^4 \tau_\alpha^2\right)$.

Lastly, we obtain from (4.28) and (4.31) that

$$\overset{\circ}{\omega}_{\mathfrak{B}''} = K^{-1}.\,dK + K^{-1}\omega_{\mathfrak{B}}\,K + \mathfrak{r}\,\mathbb{1}\,, \tag{4.36}$$

where $\mathfrak{B}'' = R_K\mathfrak{B}$ and

$$\begin{aligned} K &= f\,.\,\hat{\tau}\,,\\ &= e^{-\kappa^1}\begin{pmatrix} \tau_1 - \displaystyle\sum_{i=2}^{4}\kappa^i\tau_i & \tau_2 - \kappa^2\tau_1 - \kappa^3\tau_4 - \kappa^4\tau_3 & \tau_3 - \kappa^2\tau_4 - \kappa^3\tau_1 - \kappa^4\tau_2 & \tau_4 - \kappa^2\tau_3 - \kappa^3\tau_2 - \kappa^4\tau_1 \\ \tau_2 & \tau_1 & \tau_4 & \tau_3 \\ \tau_3 & \tau_4 & \tau_1 & \tau_2 \\ \tau_4 & \tau_3 & \tau_2 & \tau_1 \end{pmatrix}. \end{aligned} \tag{4.37}$$

Therefore, if we want a projective connection in the basis $\mathfrak{B}''$, we must begin with a connection $\bar{\omega}$ such that

$$\bar{\omega} = \omega + \mathfrak{s}\mathbb{1}\,, \tag{4.38}$$

where $\mathfrak{s}$ is a 1-form on $\mathbb{R}^4$. Then, from the relation $\bar{\omega}_{\mathfrak{B}''} = K^{-1}.\,dK + K^{-1}\omega_{\mathfrak{B}}\,K + \mathfrak{s}\,\mathbb{1}$, we deduce that

$$Tr(\bar{\omega}_{\mathfrak{B}''}) = 4\,\mathfrak{s} + d\left(\ln(N(\tau)\,e^{-4\kappa^1})\right), \tag{4.39a}$$

$$i_\xi\bar{\omega}_{\mathfrak{B}''} = \left(i_\xi\mathfrak{s} - \frac{1}{2}\right)\mathbb{1}\,. \tag{4.39b}$$

Thus, if we set

$$\mathfrak{s} \equiv -\frac{1}{4}\,d\left(\ln(N(\tau)\,e^{-4\kappa^1})\right), \tag{4.40}$$

we obtain

$$\begin{aligned} i_\xi\mathfrak{s} &= -\frac{1}{4}\,i_\xi d\left(\ln(N(\tau))\right) + i_\xi d\kappa^1\,,\\ &= -\frac{1}{4}\,Tr(\hat{\tau}^{-1}.\,i_\xi d\hat{\tau}) + 1\,,\\ &= -\frac{1}{4}\,Tr\left(\frac{1}{2}\,\mathbb{1}\right) + 1\,,\\ &= \frac{1}{2}\,. \end{aligned} \tag{4.41}$$

Therefore, $\bar{\omega}_{\mathfrak{B}''}$ is such that $Tr(\bar{\omega}_{\mathfrak{B}''}) = 0$ and $i_\xi \bar{\omega}_{\mathfrak{B}''} = 0$ but $\bar{\omega}_{\mathfrak{B}}$ is not inhomogeneous since $\mathfrak{s}$ is not.

From another point of view, if we define $\tau_{(\kappa)}$ (see Appendix I) such that

$$\hat{\tau} \equiv e^{\frac{1}{2}\,\kappa^1}\,\hat{\tau}_{(\kappa)}\,, \tag{4.42}$$

then $\tau_{(\kappa)}$ is inhomogeneous, *i.e.*, $\tau_{(\kappa)}$ depends only on the κ^i for $i = 2, 3, 4$ and the relation

$$N(\tau) = e^{2\kappa^1}\, N(\tau_{(\kappa)}) \tag{4.43}$$

holds. As a consequence, coming back to $\overset{\circ}{\omega}$, we can take $\mathfrak{r}$ such that

$$\mathfrak{r} \equiv -\frac{1}{4}\, d\left(\ln(N(\tau)\, e^{-2\kappa^1})\right) = -\frac{1}{4}\, d\left(\ln(N(\tau_{(\kappa)}))\right) \tag{4.44}$$

to obtain the following simpler expressions:

$$\overset{\circ}{\omega} \equiv \omega - \frac{1}{4}\, d\left(\ln(N(\tau_{(\kappa)}))\right) \mathbb{1}\,, \tag{4.45a}$$

$$Tr(\overset{\circ}{\omega}_{\mathfrak{B}''}) = -2d\kappa^1\,, \tag{4.45b}$$

$$i_\xi \overset{\circ}{\omega}_{\mathfrak{B}''} = -\frac{1}{2}\,\mathbb{1}\,. \tag{4.45c}$$

But then, although $\overset{\circ}{\omega}$ remains to be a projective connection it is not of Cartan type because it is not traceless.

B. The metric fields on $\mathcal{M}_{RPS}$

1. *An examples of metric field on $\mathcal{M}_{RPS}$*

Instead of the four coordinates τ_α, the local coordinates on $\mathcal{M}_{RPS}$ can also be the four variables κ^α which are inhomogeneous coordinates with respect to the projective structure. Thus, let h be a metric field on $\mathcal{M}_{RPS}$ and ∇ the covariant derivative associated with the projective Cartan connection ω given in (4.22). With respect to the coordinates κ^α, h can be written as

$$h \equiv \sum_{\alpha,\beta=1}^{4} h_{\alpha\beta}\, d\kappa^\alpha \odot d\kappa^\beta\,. \tag{4.46}$$

If we consider ∇ to be "conformally compatible" (*i.e.*, we have a Weyl geometry; see Fig. II.1, p. 14) with h, then, we must have $\nabla h \equiv 2\,h \otimes \theta$ where θ is a 1-form over $\mathcal{M}_{RPS}$ and not over $\mathbb{R}P^3$ in full generality. But, we have also (see (3.50) and corollary 1 for the covariant derivatives of 1-forms):

$$\nabla h \equiv \sum_{\alpha,\beta=1}^{4} (d\kappa^\alpha \odot d\kappa^\beta) \otimes \left(dh_{\alpha\beta} - \sum_{\gamma=1}^{4} \left(h_{\alpha\gamma}\,\omega^\gamma_{\mathfrak{B},\beta} + h_{\gamma\beta}\,\omega^\gamma_{\mathfrak{B},\alpha} \right) \right) . \tag{4.47}$$

Thus, the projective compatibility condition is satisfied when we have the following relations:

$$dh_{\alpha\beta} = \sum_{\gamma=1}^{4} \left(h_{\alpha\gamma}\,\omega^\gamma_{\mathfrak{B},\beta} + h_{\gamma\beta}\,\omega^\gamma_{\mathfrak{B},\alpha} \right) + 2\,\theta\,h_{\alpha\beta} . \tag{4.48}$$

In particular, if h is diagonal we have $h_{\alpha\alpha}\,\omega^\alpha_{\mathfrak{B},\beta} + h_{\beta\beta}\,\omega^\beta_{\mathfrak{B},\alpha} = 0$ if $\alpha \neq \beta$. Thus, in the basis $\mathfrak{B} = \{e^{\kappa^1}\tilde{\xi}, e^{\kappa^1}\,\tilde{\xi}_2,\, e^{\kappa^1}\,\tilde{\xi}_3, e^{\kappa^1}\,\tilde{\xi}_4\}$ and the dual cobasis $\mathfrak{B}^* = \{d\kappa^1, d\kappa^2, d\kappa^3, d\kappa^4\}$, we find that the projective Cartan connection ω such that

$$\omega_{\mathfrak{B}} \equiv \begin{pmatrix} 0 & d\kappa^2 & d\kappa^3 & d\kappa^4 \\ \frac{1}{c^2}\,d\kappa^2 & 0 & \not{\omega}^3_2 & -\not{\omega}^4_2 \\ \frac{1}{c^2}\,d\kappa^3 & -\not{\omega}^3_2 & 0 & \not{\omega}^4_3 \\ \frac{1}{c^2}\,d\kappa^4 & \not{\omega}^4_2 & -\not{\omega}^4_3 & 0 \end{pmatrix} , \tag{4.49}$$

preserves the 'conformal Minkowski metric:'

$$h \equiv e^{2\Theta} \left(c^2\,(d\kappa^1)^2 - (d\kappa^2)^2 - (d\kappa^3)^2 - (d\kappa^4)^2 \right) , \tag{4.50}$$

where c is a nonvanishing constant, $d\Theta \equiv \theta$ and Θ can depends on κ^1. Thus, we consider that $\mathfrak{r} = 0$ and $\overset{\circ}{\omega} \equiv \omega$ in this particular case. Also, we can associate with h the matrix H in the basis $\mathfrak{B}$ such that, obviously, H has the following form:

$$H \equiv e^{2\Theta} \begin{pmatrix} c^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{4.51}$$

In our notation, we recall that for any matrix M, we have $M \equiv (M^\beta_\alpha)$ where M^β_α is the component of M on the *column* β and the *row* α. Thus, for instance, if we define the matrix

H such that $H \equiv (H^{\alpha}_{\beta} \equiv h_{\alpha\beta})$, then, the relation (4.48) is equivalent to the expression (see also footnote (27), p. 42):

$$\nabla H = dH - H\,\omega_{\mathfrak{B}} - {}^{t}\omega_{\mathfrak{B}}\,H = 2\,d\Theta\,H\,. \tag{4.52}$$

Similarly, if we start with the connection $\overset{\circ}{\omega}_{\mathfrak{B}'}$ given by (4.28) or (4.29) with the corresponding metric field $\tilde{h}$, then, necessarily, in the corresponding dual basis $\{d\rho_1, d\rho_2, d\rho_3, d\rho_4\}$, we have

$$\tilde{h} \equiv \sum_{\alpha,\beta=1}^{4} \tilde{h}_{\alpha\beta}\,d\rho_\alpha \odot d\rho_\beta\,, \tag{4.53}$$

where, as for the metric h, the coefficients $\tilde{h}_{\alpha\beta}$ are the coefficients of the matrix $\widetilde{H} \equiv (\widetilde{H}^{\alpha}_{\beta} \equiv \tilde{h}_{\alpha\beta})$ such that

$$\widetilde{H} = {}^{t}f\,H\,f \equiv e^{2(\Theta-\kappa^1)} \begin{pmatrix} c^2 & -c^2\kappa^2 & -c^2\kappa^3 & -c^2\kappa^4 \\ -c^2\kappa^2 & c^2(\kappa^2)^2-1 & c^2\kappa^2\kappa^3 & c^2\kappa^2\kappa^4 \\ -c^2\kappa^3 & c^2\kappa^2\kappa^3 & c^2(\kappa^3)^2-1 & c^2\kappa^3\kappa^4 \\ -c^2\kappa^4 & c^2\kappa^2\kappa^4 & c^2\kappa^3\kappa^4 & c^2(\kappa^4)^2-1 \end{pmatrix}, \tag{4.54}$$

and

$$\nabla\widetilde{H} = d\widetilde{H} - \widetilde{H}\,\overset{\circ}{\omega}_{\mathfrak{B}'} - {}^{t}\overset{\circ}{\omega}_{\mathfrak{B}'}\widetilde{H} = 2\,d\Theta\,\widetilde{H}\,. \tag{4.55}$$

If we continue the computations to know the metric $\overline{H}$ associated with $\overset{\circ}{\omega}_{\mathfrak{B}''}$ defined by (4.31), then $\overline{H}$ is such that $\overline{H} \equiv \hat{\tau}\,\widetilde{H}\,\hat{\tau}$. But, the computations show that $\overline{H}$ has no vanishing coefficients on its diagonal, and thus, $\overline{H}$ differs deeply from the metric g given on $\mathcal{M}_{RPS}$, *i.e.*, the metric given by (2.1). It follows that we must begin with this latter metric and reach back to a projective connection $\not{\omega}$ using the successive changes of frames f and $\hat{\tau}$. And then after only, applying the process explained in the precedent section we can deduce a projective Cartan connection ω from $\not{\omega}$.

2. *The metric field g of the spacetime $\mathcal{M}_{RPS}$ yielded by relativistic positioning systems*

Now, we consider the metric g defined by the relation (2.1) and the connection $\overset{\circ}{\omega}_{\mathfrak{B}''}$ given by (4.31). Hence, in the present context with $\overset{\circ}{\omega}_{\mathfrak{B}''}$, we associate with the metric $g \equiv g_{\alpha\beta}\,d\tau_\alpha \odot d\tau_\beta$

given by (2.1) the matrix G such that in the basis $\{\partial_{\tau_1},\ldots,\partial_{\tau_4}\}$ we have

$$G \equiv -\frac{1}{2}\,\mathcal{V}\,\widehat{\mathfrak{H}}\,\mathcal{V} = -\frac{1}{2}\begin{pmatrix} 0 & \nu_1\nu_2 & \nu_1\nu_3 & \nu_1\nu_4 \\ \nu_2\nu_1 & 0 & \nu_2\nu_3 & \nu_2\nu_4 \\ \nu_3\nu_1 & \nu_3\nu_2 & 0 & \nu_3\nu_4 \\ \nu_4\nu_1 & \nu_4\nu_2 & \nu_4\nu_3 & 0 \end{pmatrix}, \tag{4.56a}$$

with

$$\mathcal{V} \equiv \begin{pmatrix} \nu_1 & 0 & 0 & 0 \\ 0 & \nu_2 & 0 & 0 \\ 0 & 0 & \nu_3 & 0 \\ 0 & 0 & 0 & \nu_4 \end{pmatrix}, \tag{4.56b}$$

and where the ν_α's are considered as functions of the τ_β's. Additionally, the inverse matrix is such that

$$G^{-1} \equiv \frac{1}{3}\,\mathcal{V}^{-1}\,(\mathbb{1} - 2\,\widehat{\mathfrak{H}})\,\mathcal{V}^{-1} = -\frac{2}{3}\begin{pmatrix} -\frac{2}{(\nu_1)^2} & \frac{1}{\nu_1\nu_2} & \frac{1}{\nu_1\nu_3} & \frac{1}{\nu_1\nu_4} \\ \frac{1}{\nu_2\nu_1} & -\frac{2}{(\nu_2)^2} & \frac{1}{\nu_2\nu_3} & \frac{1}{\nu_2\nu_4} \\ \frac{1}{\nu_3\nu_1} & \frac{1}{\nu_3\nu_2} & -\frac{2}{(\nu_3)^2} & \frac{1}{\nu_3\nu_4} \\ \frac{1}{\nu_4\nu_1} & \frac{1}{\nu_4\nu_2} & \frac{1}{\nu_4\nu_3} & -\frac{2}{(\nu_4)^2} \end{pmatrix}, \tag{4.56c}$$

Then, $G \equiv H$ and $\overset{\circ}{\omega}_{\mathfrak{B}''}$ must satisfy the following system of differential equations:

$$dG = G\,(\overset{\circ}{\omega}_{\mathfrak{B}''} + \theta\,\mathbb{1}) + {}^t(\overset{\circ}{\omega}_{\mathfrak{B}''} + \theta\,\mathbb{1})\,G\,. \tag{4.57}$$

where again $\theta = d\Theta$. Considering, from (4.56a), that

$$dG \equiv G\,(\mathcal{V}^{-1}d\mathcal{V}) + (\mathcal{V}^{-1}d\mathcal{V})\,G\,, \tag{4.58}$$

we obtain that the precedent relation (4.57) is equivalent to

$$\widehat{\mathfrak{H}}\,\sigma + {}^t\sigma\,\widehat{\mathfrak{H}} = 0\,, \tag{4.59}$$

where

$$\sigma \equiv \mathcal{V}\,\overset{\circ}{\omega}_{\mathfrak{B}''}\,\mathcal{V}^{-1} + \theta\,\mathbb{1} - \mathcal{V}^{-1}d\mathcal{V}\,. \tag{4.60}$$

Besides, the matrices σ such that $\widehat{\mathfrak{H}}\,\sigma + {}^t\sigma\,\widehat{\mathfrak{H}} = 0$ satisfy the following relations:

$$\sum_{\beta=1,\,\beta\neq\alpha}^{4} \sigma^\alpha_\beta = 0\,, \tag{4.61a}$$

$$\sigma^\beta_\alpha + \sigma^\alpha_\beta - \sigma^\alpha_\alpha - \sigma^\beta_\beta = 0\,, \tag{4.61b}$$

$$Tr(\sigma) = 0\,. \tag{4.61c}$$

And from these relations, we deduce in particular that

$$\sum_{\beta=1,\,\beta\neq\alpha}^{4} \sigma^\beta_\alpha = 2\,\sigma^\alpha_\alpha\,, \tag{4.61d}$$

and also that the matrix $\mathcal{V}^{-1}\sigma\,\mathcal{V}$ satisfies the following equation:

$$G\,(\mathcal{V}^{-1}\sigma\,\mathcal{V}) + {}^t(\mathcal{V}^{-1}\sigma\,\mathcal{V})\,G = 0\,. \tag{4.62}$$

The coefficients of the Euclidean connection form $\overset{\circ}{\omega}_{\mathfrak{B}''}$ satisfying the relations (4.57) or (4.59) are tremendous expressions in full generality. We will give them in the sequel only in the particular case of torsion-free Euclidean connections $\overset{\circ}{\omega}_{\mathfrak{B}''}$, *i.e.*, the Levi-Civita connection. Nevertheless, in all cases with possible torsion, $\overset{\circ}{\omega}_{\mathfrak{B}''}$ is always the sum of a torsion-free connection satisfying (4.57) and of a matrix 1-form $C \equiv \mathcal{V}^{-1}\sigma\,\mathcal{V}$ satisfying (4.62) from which torsion can originate and such that in the basis $\{\partial_{\tau_1},\ldots,\partial_{\tau_4}\}$ and from the relations (4.61) the matrix σ has the following form:

$$\sigma \equiv \begin{pmatrix} 2(\varrho^2_1+\varrho^3_1)+\varrho^1_3+\varrho^2_4+\varrho^3_4+\varrho^1_2 & 2\varrho^2_1 & 2\varrho^3_1 & 2(\varrho^1_3+\varrho^3_1+\varrho^2_4+\varrho^1_2+\varrho^2_1+\varrho^3_4) \\ 2\varrho^1_2 & \varrho^1_2-\varrho^1_3-\varrho^2_4-\varrho^3_4-2\varrho^3_1 & -2(\varrho^3_1+\varrho^3_4) & -2(\varrho^3_1+\varrho^1_3+\varrho^2_4) \\ 2\varrho^1_3 & -2(\varrho^2_1+\varrho^2_4) & \varrho^1_3-\varrho^2_4-\varrho^3_4-\varrho^1_2-2\varrho^2_1 & -2(\varrho^2_1+\varrho^1_2+\varrho^3_4) \\ -2(\varrho^1_2+\varrho^1_3) & 2\varrho^2_4 & 2\varrho^3_4 & \varrho^2_4+\varrho^3_4-\varrho^1_2-\varrho^1_3 \end{pmatrix}, \tag{4.63}$$

where $Tr(\sigma) = 0$ and where the ϱ^i_j are 1-forms defined on $\mathcal{M}_{RPS}$ and not $\mathbb{R}P^3$ (*i.e.*, depending on the four coordinates κ^α in full generality, three of them being inhomogeneous coordinates).

Also, note that σ cannot be a nonvanishing diagonal matrix.

Thus, we have also that

$$\overset{\circ}{\omega}_{\mathfrak{B}''} = \mathcal{V}^{-1}d\mathcal{V} + \mathcal{V}^{-1}\sigma\,\mathcal{V} - \theta\,\mathbb{1}\,, \tag{4.64}$$

and we consider $\overset{\circ}{\omega}_{\mathfrak{B}''} \equiv \Gamma_{\mathfrak{B}''}$ to be the *Levi-Civita connection* of the metric g (see Appendix K) or a connection with, possibly, torsion but with no *nonmetricity*. Therefore, in full generality, $\overset{\circ}{\omega}_{\mathfrak{B}''}$ decomposes in three main parts: the so-called *Weitzenböck's flat connection*

$$\mathcal{V}^{-1} d\mathcal{V}\,, \tag{4.65}$$

and the mixed term

$$\mathcal{V}^{-1} \sigma\, \mathcal{V} - \theta\,, \tag{4.66}$$

which contains a part of the Levi-Civita connection tensor and, eventually, the *contorsion tensor* but also other sorts of connections (see [ABP03, Iti04, Iti07, Bel08, Iti13]). Besides, we set $C \equiv \mathcal{V}^{-1} \sigma\, \mathcal{V}$ and then, the relation

$$G \,.\, C + {}^{t}C \,.\, G = 0\,. \tag{4.67}$$

holds. Also, from (4.35a) and $Tr(\sigma) = 0$, we deduce that $d \ln(N(\tau)\, e^{-4\kappa^1}) + 4\,\mathfrak{r} = Tr(\mathcal{V}^{-1} d\mathcal{V}) - 4\theta$, and therefore, with the definition of $\tau_{(\kappa)}$ given by (4.43) and $\theta = d\Theta$, the four functions $\nu_\alpha(\tau)$ must satisfy the relation

$$e^{2\kappa^1} \prod_{\alpha=1}^{4} \vartheta_\alpha(\tau) = e^{4\mathfrak{t}}\, N(\tau_{(\kappa)})\,, \tag{4.68}$$

where $\vartheta_\alpha \equiv e^{-\Theta}\, \nu_\alpha$ and where $\mathfrak{t}$ depending only on the κ_i $(i = 2, 3, 4)$ is such that

$$\mathfrak{r} \equiv d\mathfrak{t}\,. \tag{4.69}$$

Moreover, from the relation (4.35b), *i.e.*, $i_\xi \overset{\circ}{\omega}_{\mathfrak{B}''} = -\frac{1}{2}\,\mathbb{1}$, we deduce that

$$\mathcal{V}^{-1}\, i_\xi d\mathcal{V} + i_\xi \sigma = \Big(i_\xi d\Theta - \tfrac{1}{2}\Big)\, \mathbb{1}\,. \tag{4.70}$$

But also, $\mathcal{V}^{-1} d\mathcal{V}$ is diagonal, and therefore, the non-diagonal terms of $i_\xi \sigma$ must vanish. And, in addition, the diagonal terms of σ are linear combinations of the non-diagonal terms. Thus, we conclude that the relations

$$i_\xi \sigma = 0\,, \tag{4.71a}$$

$$i_\xi d\vartheta_\alpha = -\frac{1}{2}\, \vartheta_\alpha\,, \tag{4.71b}$$

must hold. Furthermore, from the definition of ξ given by (4.20), we have

$$\xi \equiv \sum_{\alpha=1}^{4} \rho_\alpha \frac{\partial}{\partial \rho_\alpha}\,, \tag{4.72}$$

and thus, from the Euler's theorem on homogeneous functions, the functions ϑ_α and the matrix σ must be, respectively, homogeneous functions of degree $-\frac{1}{2}$ and a homogeneous matrix of degree zero, *i.e.*, horizontal, with respect to the variables ρ_α. Or, equivalently, *the functions ϑ_α and the matrix σ must be, respectively, homogeneous functions of degree -1 and a homogeneous matrix of degree zero, i.e., horizontal, with respect to the four time stamps τ_α.*

Besides, we have the general relation:

$$|g| \equiv -\det g = \frac{3}{16}\left(\prod^{4} \nu_\alpha\right)^2 > 0\,. \tag{4.73}$$

Also, by definition, $\mathcal{M}_{RPS}$ is a *Weyl integrable manifold* if and only if θ is an exact 1-form.[32] Thus, $\mathcal{M}_{RPS}$ is an integrable Weyl manifold. Moreover, we can take

$$\mathfrak{r} \equiv 0\,, \tag{4.74a}$$

and

$$\Theta \equiv \frac{1}{2}\,\kappa^1\,, \tag{4.74b}$$

and then, $\prod_{\alpha=1}^{4} \nu_\alpha(\tau) = N(\tau_{(\kappa)})$ and

$$Tr(\overset{\circ}{\omega}_{\mathfrak{B}''}) = Tr(\omega_{\mathfrak{B}''}) = d\ln(N(\tau_{(\kappa)})) - 2d\kappa^1\,. \tag{4.75}$$

As a result, we obtain the first following constraint to have a projective Cartan connection $\omega_{\mathfrak{B}}$:

$$|g| = \frac{3}{16}\,N(\tau_{(\kappa)})^2 = \frac{3}{16}\,\frac{N(\tau)^2}{\left(\sum_{\alpha=1}^{4}(\tau_\alpha)^2\right)^4}\,. \tag{4.76}$$

[32] Actually, a Weyl connection $\widetilde{\Gamma}$ associated with a metric $\tilde{g}$ is, usually, a Levi-Civita connection. Indeed, $\widetilde{\Gamma}$ is a Levi-Civita connection defined from another Levi-Civita connection Γ associated with another metric g conformally defined by $g \equiv e^{-2\varphi}\,\tilde{g}$. And then, we obtain: $\widetilde{\Gamma}^\alpha_{\beta\gamma} \equiv \Gamma^\alpha_{\beta\gamma} + \frac{1}{2}\,g^{\alpha\mu}\left(g_{\mu\beta}\,\psi_\gamma + g_{\mu\gamma}\,\psi_\beta - g_{\beta\gamma}\,\psi_\mu\right)$ where $\psi \equiv \psi_\alpha\,dx^\alpha = d\varphi$. The relation between $\widetilde{\Gamma}$ and Γ is only related to a change of torsion-free Riemannian structure coming from a change of Riemannian manifold due to two different but conformally equivalent metrics. Because ψ is exact, we have a so-called '*Weyl integrable manifold*' (see for instance [RFNP12]) and *not-integrable* otherwise. In that latter case, because the Levi-Civita connection is the unique torsion-free connection on any given open neighborhood in a Riemannian manifold $\mathcal{M}_{RPS}$, ψ can only be closed on $\mathcal{M}_{RPS}$; and this expresses only an equivalence of Riemannian structures. If not, *i.e.*, $d\psi \neq 0$ on $\mathcal{M}_{RPS}$, then, $\widetilde{\Gamma}$ cannot be a connection form and the expression for $\widetilde{\Gamma}$ with respect to Γ is therefore void of meaning.

Nevertheless, it remains four other constraints due to the "values" $\omega^0_{\mathfrak{B},0} = 0$ and $\omega^i_{\mathfrak{B},0} = d\kappa^i$.

Lastly, to summarize this section, we notice that we began with a given projective connection $\omega_{\mathfrak{B}}$ and successive changes of frames defining the relationship between homogeneous and inhomegeneous coordinates, and then, we deduced successive metric fields compatible with the successive connections. But, we did not deduce completely the metric field on $\mathcal{M}_{RPS}$, *i.e.*, the functions $\nu_\alpha(\tau)$ from the definition of $\omega_{\mathfrak{B}}$. Actually, it is somewhat easy noticing that $\omega_{\mathfrak{B}''}$ refers to a matrix $G \equiv G_{\mathfrak{B}''}$ given by (4.56a) and defined essentially by $\mathcal{V}$ and $\widehat{\mathfrak{H}}$. Then, $\omega_{\mathfrak{B}}$ refers to a metric $G_{\mathfrak{B}}$ such that

$$G_{\mathfrak{B}} = -\frac{1}{2}\, {}^tU\, \widehat{\mathfrak{H}}\, U\,, \tag{4.77a}$$

where

$$U \equiv \mathcal{V}\, \hat{\tau}^{-1}\, f^{-1}\,. \tag{4.77b}$$

And thus, we have also

$$G_{\mathfrak{B}''} = {}^tK\, G_{\mathfrak{B}}\, K\,. \tag{4.78}$$

Moreover, we deduce the important result from (4.70), (4.71a) and (4.74): $i_\xi d\nu_\alpha = 0$ for all α since $\mathcal{V}$ is diagonal. And then, we have the following:

Lemma 3. *The 'experienced spacetime'* $\mathcal{M}_{RPS}$ *admits a projective connection* ω *only if the four functions* ν_α *equal functions depending only on the three inhomogeneous coordinates* κ^i $(i = 2, 3, 4)$ *and*

$$\prod_{\alpha=1}^{4} \nu_\alpha(\tau) = N(\tau_{(\kappa)})\,, \tag{4.79}$$

where $\tau_{(\kappa)}$ *is given in Appendix* I.

Also, to summarize, we have the following formulas:

$$\mathfrak{r} = 0, \tag{4.80a}$$

$$\overset{\circ}{\omega} = \omega, \tag{4.80b}$$

$$i_\xi \omega_{\mathfrak{B}} = 0, \tag{4.80c}$$

$$Tr(\omega_{\mathfrak{B}}) = 0, \tag{4.80d}$$

$$\Theta = \frac{1}{2}\,\kappa^1, \tag{4.80e}$$

$$\theta = d\Theta, \tag{4.80f}$$

$$\tau_{(\kappa)} \equiv e^{-\frac{1}{2}\,\kappa^1}\,\tau, \tag{4.80g}$$

$$N(\tau_{(\kappa)}) = e^{-2\kappa^1}\,N(\tau), \tag{4.80h}$$

$$Tr(\omega_{\mathfrak{B}''}) = d\ln(N(\tau_{(\kappa)})\,e^{-2\kappa^1}), \tag{4.80i}$$

$$i_\xi \omega_{\mathfrak{B}''} = -\frac{1}{2}\,\mathbb{1}, \tag{4.80j}$$

$$\omega_{\mathfrak{B}''} = K^{-1}\,dK + K^{-1}\,\omega_{\mathfrak{B}}\,K\,, \tag{4.80k}$$

$$\omega_{\mathfrak{B}} = K\,\omega_{\mathfrak{B}''}\,K^{-1} - dK\,.\,K^{-1}, \tag{4.80l}$$

$$G_{\mathfrak{B}''} = {}^t\!K\,G_{\mathfrak{B}}\,K, \tag{4.80m}$$

$$dG_{\mathfrak{B}''} - G_{\mathfrak{B}''}\,\omega_{\mathfrak{B}''} - {}^t\omega_{\mathfrak{B}''}\,G_{\mathfrak{B}''} = 2\,\theta\,G_{\mathfrak{B}''}, \tag{4.80n}$$

$$dG_{\mathfrak{B}} - G_{\mathfrak{B}}\,\omega_{\mathfrak{B}} - {}^t\omega_{\mathfrak{B}}\,G_{\mathfrak{B}} = 2\,\theta\,G_{\mathfrak{B}}\,. \tag{4.80o}$$

And, moreover, we have also

$$\omega^1_{\mathfrak{B},1} = 0\,, \tag{4.81a}$$

$$\omega^i_{\mathfrak{B},1} = d\kappa^i\,. \tag{4.81b}$$

Remark 12. *These last two formulas or constraints yield tremendous formulas, constraints or systems of PDEs which must be satisfied by $\omega_{\mathfrak{B}''}$. The last one is not always needed but it must be satisfied if we want $\omega_{\mathfrak{B}}$ to provide a* $\mathfrak{B}$-complete projective Cartan connection. *In any case, only the first formula is required to have a projective Cartan connection $\omega_{\mathfrak{B}}$.*

V. THE PROJECTIVE CURVATURE 2-FORM Ω

A. The general case on $\mathbb{R}P^n$

Using the notations of section III C 1 (p.32) again and considering a given $n+1$-dimensional frame field $\mathfrak{B} \equiv \{\mathfrak{v}_0, \mathfrak{v}_1, \ldots, \mathfrak{v}_n\}$ of $T\mathbb{R}^{n+1}$ over $\mathbb{R}P^n$, then, the Riemannian curvature 2-form $\not\Omega$ over $\mathbb{R}P^n$ obtained from a given projective connection $\not\omega_{\mathfrak{B}}$ over $\mathbb{R}P^n$ associated with the covariant derivative $\not\nabla$ is defined by the following formula:

$$\not\Omega = d\not\omega_{\mathfrak{B}} + \not\omega_{\mathfrak{B}} \wedge \not\omega_{\mathfrak{B}} \,. \tag{5.1}$$

Note, that $\not\omega_{\mathfrak{B}}$ depends on the field $\mathfrak{B}$ (a section of the frame bundle) since a connection is a field defined on the frame bundle contrary to $\not\Omega$ which is a field on $M \equiv \mathbb{R}^{n+1}$, and thus, $\not\Omega$ does not depend on any frame field $\mathfrak{B}$, $\mathfrak{B}'$..., *i.e.*, $\not\Omega = d\not\omega_{\mathfrak{B}} + \not\omega_{\mathfrak{B}} \wedge \not\omega_{\mathfrak{B}} = d\not\omega_{\mathfrak{B}'} + \not\omega_{\mathfrak{B}'} \wedge \not\omega_{\mathfrak{B}'} = \ldots$ contrary, obviously, to its components. As a result, we can expressed the components of $\not\omega_{\mathfrak{B}}$ in another frame field $\mathfrak{B}'$ and we obtain the components $\not\omega^{\beta}_{\mathfrak{B},\mathfrak{B}',\alpha}$. Then, we consider that $\not\omega^{\beta}_{\mathfrak{B},\alpha} \equiv \not\omega^{\beta}_{\mathfrak{B},\mathfrak{B},\alpha}$, *i.e.*, we evaluate the components of $\not\omega_{\mathfrak{B}}$ in the same frame $\mathfrak{B}$. In other words, $\not\omega_{\mathfrak{B}}$ is affine with respect to a change of frame field, *i.e.*, we have an affine (gauge) transformation passing from $\not\omega_{\mathfrak{B}}$ to $\not\omega_{\mathfrak{B}'}$ (we have for example $\not\omega_{\mathfrak{B}'} = K^{-1}\, dK + K^{-1} \omega_{\mathfrak{B}}\, K$ where $\mathfrak{B}' = K\,\mathfrak{B}$). But, it is a tensor with respect to changes of frames for its components $\not\omega^{\beta}_{\mathfrak{B},\mathfrak{B}',\alpha}$, *i.e.*, we pass from $\not\omega^{\beta}_{\mathfrak{B},\mathfrak{B}',\alpha}$ to $\not\omega^{\nu}_{\mathfrak{B},\mathfrak{B}'',\mu}$ by a linear transformation (we can expressed the formula $\not\omega_{\mathfrak{B}'} = K^{-1}\, dK + K^{-1} \omega_{\mathfrak{B}}\, K$ in a basis $\mathfrak{B}''$ and then $\not\omega^{\beta}_{\mathfrak{B}',\mathfrak{B}'',\alpha} = K^{-1,\mu}_{\mathfrak{B}'',\alpha}\; dK^{\beta}_{\mathfrak{B}'',\mu} + K^{-1,\mu}_{\mathfrak{B}'',\alpha}\; \omega^{\nu}_{\mathfrak{B},\mathfrak{B}'',\mu}\, K^{\beta}_{\mathfrak{B}'',\nu}$ where the components are expressed with respect to the vectors of $\mathfrak{B}''$. Then, passing from $\mathfrak{B}''$ to $\mathfrak{B}'''$ the formula above is linearly transformed in an equivalent formula with components in the frame $\mathfrak{B}'''$). We use this convention of notations in the whole of the present document for any connection ω. Sometimes, if a basis $\mathfrak{B}$ is not need to be specified because implicit in the definition of a given connection ω then we write simply $\omega_{\mathfrak{B}} \equiv \omega$ in accordance with the rule $\omega^{\beta}_{\mathfrak{B},\alpha} \equiv \omega^{\beta}_{\mathfrak{B},\mathfrak{B},\alpha}$, *i.e.*, a connection is never defined in an abstract way, contrary to its gauge transformations, but in an explicit given frame giving its components. Hence, we always need to use indices and a frame field $\mathfrak{B}$ in any explicit definition of a connection. We can write also $\not\Omega = d\not\omega + \not\omega \wedge \not\omega$ without any specified basis. In this case, we mean that $\not\Omega$ is considered as a field on the frame bundle of $M \equiv \mathbb{R}^{n+1}$ but constant with respect to any basis $\mathfrak{B}$, *i.e.*, independent on any basis $\mathfrak{B}$ but only on its base point in M. In other words, $\not\Omega$ is gauge-invariant.

Then, with these conventions of notations, in the basis $\mathfrak{B}$ and its dual basis $\mathfrak{B}^* \equiv \{\mathfrak{v}^{*0}, \mathfrak{v}^{*1}, \dots, \mathfrak{v}^{*n}\}$, we can write also the defining formulas for the 2-form components of $\not\Omega$ (see footnote 27, p.42 for the conventions of matrix representations and indices):

$$\not\Omega^{\alpha}_{\mathfrak{B},\beta} = d\not\omega^{\alpha}_{\mathfrak{B},\beta} + \sum_{\gamma=0}^{n} \not\omega^{\alpha}_{\mathfrak{B},\gamma} \wedge \not\omega^{\gamma}_{\mathfrak{B},\beta}\,. \tag{5.2}$$

Moreover, $\not\Omega$ is nothing more [Hel09] than the Riemann curvature tensor $\not R$ such that

$$\not R(\mathfrak{v},\mathfrak{w})\mathfrak{u} \equiv \not\nabla_{\mathfrak{v}}(\not\nabla_{\mathfrak{w}}\mathfrak{u}) - \not\nabla_{\mathfrak{w}}(\not\nabla_{\mathfrak{v}}\mathfrak{u}) - \not\nabla_{[\mathfrak{v},\mathfrak{w}]}\mathfrak{u} = \not\Omega(\mathfrak{v},\mathfrak{w})\mathfrak{u}\,. \tag{5.3}$$

From this relation, defining $\not\Omega$ from $\not\nabla$, we see in particular that $\not\Omega$ is a tensor with respect to $GL(n+1,\mathbb{R})$, and then, its definition is not depending on the frame field $\mathfrak{B}$ contrary to $\not\omega_{\mathfrak{B}}$, although its components in a given basis $\mathfrak{B}$ depend obviously on.

With indices, we have also the following well-known usual expressions for the ($\mathfrak{B}$-dependent) components of $\not\Omega$: $\not\Omega^{\alpha}_{\beta,\varkappa\varsigma} \equiv \not\Omega^{\alpha}_{\beta}(\mathfrak{v}_{\varkappa},\mathfrak{v}_{\varsigma})$. Besides, we have seen (see Theorem 5, p. 61) that a projective Cartan covariant derivative ∇ associated with ω can be obtained from any covariant derivative $\not\nabla$ by subtracting $\pi(\not\nabla\xi)\mathbb{1}$ (*i.e.*, $\not\omega^{0}_{\mathfrak{B},0}\mathbb{1}$ if $\xi \equiv \mathfrak{v}_0$) to $\not\nabla$: $\nabla \equiv \not\nabla - \pi(\not\nabla\xi)\mathbb{1}$. Now, let $\mathfrak{u}$, $\mathfrak{v}$ and $\mathfrak{w}$ be any three vector fields in $\chi(\mathbb{R}P^n)$ and $\widetilde{\Omega}$ such that

$$\widetilde{\Omega}(\mathfrak{v},\mathfrak{w})\mathfrak{u} \equiv \nabla_{\mathfrak{v}}(\nabla_{\mathfrak{w}}\mathfrak{u}) - \nabla_{\mathfrak{w}}(\nabla_{\mathfrak{v}}\mathfrak{u}) - \nabla_{[\mathfrak{v},\mathfrak{w}]}\mathfrak{u} \tag{5.4}$$

or equivalently, such that

$$\widetilde{\Omega} \equiv d\omega + \omega \wedge \omega\,. \tag{5.5}$$

Then, we can define the relation between $\not\Omega$ and $\widetilde{\Omega}$ as follows from the following first results set:

$$\begin{aligned} \not\nabla_{\mathfrak{u}}\not\nabla_{\mathfrak{v}}\mathfrak{w} &\equiv (\nabla_{\mathfrak{u}} + \pi(\not\nabla_{\mathfrak{u}}\xi)\mathbb{1})(\nabla_{\mathfrak{v}}\mathfrak{w} + \pi(\not\nabla_{\mathfrak{v}}\xi)\mathfrak{w}) \\ &= \nabla_{\mathfrak{u}}\nabla_{\mathfrak{v}}\mathfrak{w} + \pi(\not\nabla_{\mathfrak{u}}\xi)\nabla_{\mathfrak{v}}\mathfrak{w} + \pi(\not\nabla_{\mathfrak{v}}\xi)\nabla_{\mathfrak{u}}\mathfrak{w} + \pi(\not\nabla_{\mathfrak{u}}\xi)\,\pi(\not\nabla_{\mathfrak{v}}\xi)\,\mathfrak{w} + i_{\mathfrak{u}}d\big(\pi(\not\nabla_{\mathfrak{v}}\xi)\big)\mathfrak{w}\,. \end{aligned} \tag{5.6}$$

And then, we have also that

$$\begin{aligned} \not\Omega(\mathfrak{u},\mathfrak{v})\mathfrak{w} &\equiv \not\nabla_{[\mathfrak{u}}\not\nabla_{\mathfrak{v}]}\mathfrak{w} - \not\nabla_{[\mathfrak{u},\mathfrak{v}]}\mathfrak{w} \\ &= \nabla_{[\mathfrak{u}}\nabla_{\mathfrak{v}]}\mathfrak{w} - \nabla_{[\mathfrak{u},\mathfrak{v}]}\mathfrak{w} + i_{\mathfrak{u}}d\big(\pi(\not\nabla_{\mathfrak{v}}\xi)\big)\mathfrak{w} - i_{\mathfrak{v}}d\big(\pi(\not\nabla_{\mathfrak{u}}\xi)\big)\mathfrak{w} - \pi(\not\nabla_{[\mathfrak{u},\mathfrak{v}]}\xi)\mathfrak{w} \\ &= \widetilde{\Omega}(\mathfrak{u},\mathfrak{v})\mathfrak{w} + i_{\mathfrak{u}}d\big(\pi(\not\nabla_{\mathfrak{v}}\xi)\big)\mathfrak{w} - i_{\mathfrak{v}}d\big(\pi(\not\nabla_{\mathfrak{u}}\xi)\big)\mathfrak{w} - \pi(\not\nabla_{[\mathfrak{u},\mathfrak{v}]}\xi)\mathfrak{w}\,. \end{aligned} \tag{5.7}$$

Then, we denote by $\not\chi$ the 1-form over $\mathbb{R}P^n$ such that

$$\not\chi(\mathfrak{u}) \equiv \pi(\not\nabla_{\mathfrak{u}}\xi), \tag{5.8}$$

and we deduce that

$$\widetilde{\Omega} = \not\Omega - d\not\chi\,\mathbb{1} \qquad \Longleftrightarrow \qquad \not\Omega = d\not\chi\,\mathbb{1} + d\omega + \omega\wedge\omega\,. \tag{5.9}$$

Obviously, we would obtain more rapidly the same result starting with

$$\omega \equiv \not\omega - \not\chi\mathbb{1}\,. \tag{5.10}$$

Thus, while ∇ is projectively equivalent to $\not\nabla$, *i.e.*, $\nabla \sim_P \not\nabla$ (or $\omega \sim_P \not\omega$), and because each curvature tensor $\not\Omega$ or $\widetilde{\Omega}$ is a tensor with respect to $GL(n+1,\mathbb{R})$ (and thus, with respect to $\mathbb{R}\times Stab([\xi])$ also), the relation (5.9) shows that the formulas (5.1) or (5.5) give tensors not intrinsic to the projective structure, *i.e.*, independent on each projective connection ω belonging to the same projective equivalence class. In addition, we would have obtained a similar result, *viz.*, $\widetilde{\Omega} = \Omega' - d\mathfrak{r}\,\mathbb{1}$, starting with ∇' defined in (3.37) instead of $\not\nabla$, and thus, such that $\nabla' \equiv \nabla + \mathfrak{r}\mathbb{1}$, where $\mathfrak{r}$ is a horizontal 1-form over $\mathbb{R}P^n$.

Actually, $i_{\mathfrak{u}}\not\Omega$ is like a projective connection 1-form, and thus, it is projectively equivalent to $i_{\mathfrak{u}}\not\Omega + \mathfrak{R}(\mathfrak{u})\mathbb{1}$ where $\mathfrak{R}$ is a horizontal 2-form over $\mathbb{R}P^n$ (*i.e.*, inhomogeneous).[33] Hence, we set the following definition of the projective curvature Ω defined from $\not\Omega$:

Definition 14. *Let $\mathfrak{u}$ and $\mathfrak{v}$ be any vector fields in $\chi(\mathbb{R}P^n)$. In addition, let $\not\nabla$ be a projective covariant derivative defined over $\mathbb{R}P^n$, and $\not\Omega$ its corresponding Riemannian curvature 2-form defined by the relations* (5.3) *(or* (5.1) *as well); Then, $\not\nabla$ defines a projective Cartan curvature Ω associated with the projective Cartan covariant derivative $\nabla_{\mathfrak{u}}\mathfrak{v} \equiv \not\nabla_{\mathfrak{u}}\mathfrak{v} - \pi(\not\nabla_{\mathfrak{u}}\xi)\mathfrak{v}$ such that Ω is defined by the relation:*

$$\Omega(\mathfrak{u},\mathfrak{v}) \equiv \not\Omega(\mathfrak{u},\mathfrak{v}) - \pi(\not\Omega(\mathfrak{u},\mathfrak{v})\,.\,\xi)\mathbb{1}\,, \tag{5.11a}$$

or, equivalently (from the relation (5.9)*):*

$$\Omega \equiv d\omega + \omega\wedge\omega - \pi\Big((\omega\wedge\omega)\xi\Big)\mathbb{1}\,. \tag{5.11b}$$

[33] Besides, this indicates that we have sets of recursive connections starting from a given connection: $\omega_0 \longrightarrow \omega_1 \equiv i_{\mathfrak{u}}\Omega_0 \longrightarrow \omega_2 \equiv i_{\mathfrak{u}}\Omega_1 \longrightarrow \ldots$

where $\omega \equiv \not\omega - \pi(\not\nabla\xi)\mathbb{1}$ *is the projective Cartan connection associated with* ∇.

Also, it is important to note that the definition of Ω differs strongly from the one given for $\widetilde{\Omega}$. Indeed, in particular, we have

$$\Omega = \widetilde{\Omega} - \pi\left(\widetilde{\Omega}\,.\,\xi\right)\mathbb{1}\,. \tag{5.12}$$

Moreover, we have:

$$\pi(\Omega\,.\,\xi) \equiv \Omega^0_{\mathfrak{B},0} = 0\,. \tag{5.13}$$

Let us note that $\pi\Big((d\omega)\xi\Big) = 0$ from $\pi(\omega\,.\,\xi) = 0$. Then, it follows the important property:

Lemma 4. *Let* ω *and* ω' *be two projective connections, i.e., Euclidean connections over* $\mathbb{R}P^n$*, such that*

$$\omega' \sim_P \omega\,,$$

and Ω *and* Ω' *their corresponding projective curvatures defined by the formula* (5.11b)*, then, we have*

$$\Omega' = \Omega + d\mathfrak{r}_1 \oplus d\mathfrak{r}_n\mathbb{1}_n + d\tilde{\mathfrak{s}} + \tilde{\mathfrak{s}}\wedge\tilde{\mathfrak{s}} + \omega\wedge\tilde{\mathfrak{s}} + \tilde{\mathfrak{s}}\wedge\omega - \pi\Big((\tilde{\mathfrak{s}}\wedge\omega)\,.\,\xi\Big)\mathbb{1}\,, \tag{5.14}$$

if $\omega' = \omega + \mathfrak{A}$ *where* $\mathfrak{A} = \mathfrak{r}\mathbb{1} + \mathfrak{s} \equiv \mathfrak{r}_1 \oplus \mathfrak{r}_n\mathbb{1}_n + \tilde{\mathfrak{s}} \in \Gamma_n((\mathbb{R}\oplus stab([\xi]))\otimes T^*\mathbb{R}P^n)$ *and* $1 \oplus \mathbb{1}_n \equiv \mathbb{1}_{n+1} \equiv \mathbb{1}$ *whatever are the horizontal 1-forms* $\mathfrak{r}_1$*,* $\mathfrak{r}_n$ *and* $\tilde{\mathfrak{s}}$ *such that*

$$\tilde{\mathfrak{s}}\,.\,\xi = 0. \tag{5.15}$$

Then, we say that Ω' *is* projectively equivalent *to* Ω*, i.e.,* $\Omega' \sim_P \Omega$*.*[34]

The condition (5.15) means that for any $\tilde{\mathfrak{s}} \in stab([\xi])$, we consider the Lie algebra decomposition $stab([\xi])) \simeq \mathbb{R}\oplus(\mathbb{R}^n \rtimes sl(n,\mathbb{R})) \subset \mathbb{R}\oplus stab([\xi]))$. And then, $\mathfrak{s} \equiv \mathfrak{r}_n\mathbb{1}_n + \tilde{\mathfrak{s}}$ where $\tilde{\mathfrak{s}} \in \mathbb{R}^n \rtimes sl(n,\mathbb{R})$ and thus, $\tilde{\mathfrak{s}}\,.\,\xi = 0$. Also, applying this decomposition to the definition (3.37) of the projective equivalence between projective connections restricts the set of map $\mathfrak{s}$ to those verifying the relation (5.15).

[34] In this definition, the projective special linear group $PSL(n+1,\mathbb{R})$ appears in the decomposition of the projective connections while the projective linear group $PGL(n+1,\mathbb{R})$ remains the group of the projective structure. In other words, although $PSL(n+1,\mathbb{R})$ intervenes in the projective equivalence, it cannot be used as the group of the projective structure contrary to what would be suggested in [Sha97, §8, p.333] for instance.

Proof. Obvious from the definition (5.11b) and because ω and $\omega' = \omega + \mathfrak{A}$ differ by a 1-form $\mathfrak{A} \equiv \mathfrak{r}_1 \oplus \mathfrak{r}_n \mathbb{1}_n + \tilde{\mathfrak{s}} \in \Gamma_n((\mathbb{R} \oplus stab([\xi])) \otimes T^*\mathbb{R}P^n)$. Indeed, we obtain

$$\begin{aligned}\Omega' &= d\omega' + \omega' \wedge \omega' - \pi\Big((\omega' \wedge \omega') \,.\, \xi\Big)\mathbb{1} \\ &= d\omega + \omega \wedge \omega + d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A} + \omega \wedge \mathfrak{A} + \mathfrak{A} \wedge \omega \\ &\qquad - \pi\Big((\omega \wedge \omega + \mathfrak{A} \wedge \mathfrak{A} + \omega \wedge \mathfrak{A} + \mathfrak{A} \wedge \omega) \,.\, \xi\Big)\mathbb{1} \\ &= \Omega + d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A} + \omega \wedge \mathfrak{A} + \mathfrak{A} \wedge \omega - \pi\Big((\mathfrak{A} \wedge \mathfrak{A} + \omega \wedge \mathfrak{A} + \mathfrak{A} \wedge \omega) \,.\, \xi\Big)\mathbb{1} \,.\end{aligned}$$

But, $\tilde{\mathfrak{s}} \,.\, \xi = 0$, and thus, we obtain $\mathfrak{A}\,\xi = \mathfrak{r}_1 \,.\, \xi \equiv (\mathfrak{r}_1 \mathbb{1}) \,.\, \xi$. Therefore, because $\mathfrak{r}\mathbb{1} \wedge \rho = -\rho \wedge \mathfrak{r}\mathbb{1}$ for any matrix-valued 1-form ρ, we have also

$$\pi\Big((\mathfrak{A} \wedge \mathfrak{A} + \omega \wedge \mathfrak{A} + \mathfrak{A} \wedge \omega) \,.\, \xi\Big) = \pi\Big((\tilde{\mathfrak{s}} \wedge \omega) \,.\, \xi\Big) \,.$$

Consequently, we obtain

$$\Omega' = \Omega + d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A} + \omega \wedge \mathfrak{A} + \mathfrak{A} \wedge \omega - \pi\Big((\tilde{\mathfrak{s}} \wedge \omega) \,.\, \xi\Big)\mathbb{1} \,.$$

And then, we deduce the relation (5.14). □

Corollary 2. *Let Ω' and Ω be two equivalent projective curvatures given by the formula* (5.11b) *where the ω's are projective connections, i.e., $\Omega' \sim_P \Omega$ then, we have*

$$\Omega' \,.\, \xi = \Omega \,.\, \xi + d\mathfrak{r}_1\, \xi + \underline{(\tilde{\mathfrak{s}} \wedge \omega) \,.\, \xi} \,, \tag{5.16}$$

where $\tilde{\mathfrak{s}}$ is such that $\tilde{\mathfrak{s}} \,.\, \xi = 0$.

Proof. Indeed, we have $(d\tilde{\mathfrak{s}} + \tilde{\mathfrak{s}} \wedge \tilde{\mathfrak{s}} + \omega \wedge \tilde{\mathfrak{s}}) \,.\, \xi = 0$ and, moreover, $(\tilde{\mathfrak{s}} \wedge \omega) \,.\, \xi = \sum_{i=1}^n \tilde{\mathfrak{s}} \wedge \omega^i_{\mathfrak{B},0}\, \mathfrak{v}_i = \sum_{i=1}^n \tilde{\mathfrak{s}}(\mathfrak{v}_i) \wedge \omega^i_{\mathfrak{B},0} = \sum_{i,j=1}^n (\tilde{\mathfrak{s}}^j_{\mathfrak{B},i} \wedge \omega^i_{\mathfrak{B},0})\mathfrak{v}_j + \pi\Big((\tilde{\mathfrak{s}} \wedge \omega)\xi\Big)\xi$; hence, the result. □

Also, we can deduce the following definition:

Definition 15. *Let ω be any projective Cartan connection, then, ω defines a projective Cartan curvature Ω such that*

$$\Omega \equiv d\omega + \omega \wedge \omega - \pi\Big((\omega \wedge \omega)\xi\Big)\mathbb{1} \,. \tag{5.17}$$

Additionally, we have the following relation on the traces.[35]

Property 1. *Let ω be a projective Cartan connection and Ω its corresponding projective curvature, then, we have*

$$Tr(\Omega) = -(n+1)\,\pi\Big((\omega\wedge\omega)\xi\Big)\,. \tag{5.18}$$

Proof. Obvious from $Tr(\omega)=0$ and $Tr(\omega\wedge\omega)=0$. ☐

Note that if there exists a basis $\mathfrak{B}$ such that $\not{\omega}^0_{\mathfrak{B},0}=0$, $Tr(\omega)=Tr(\not{\omega})=0$ then $Tr(\not{R})=0$ but we may have $Tr(\Omega)\neq 0$. Then, we have the following fundamental example due to Cartan.

Example 1. In the frame basis $\mathfrak{B}\equiv\{\mathfrak{v}_0\equiv\xi,\mathfrak{v}_1\equiv\tau_0\,\xi_1,\ldots,\mathfrak{v}_n\equiv\tau_0\,\xi_n\}$ used to express the projective Cartan connection ω with the formula (3.63), we have, actually, $\omega\equiv\not{\omega}$ and then $\pi(\not{\nabla}\xi)\equiv\omega^0_{\mathfrak{B},0}\equiv 0$. Thus, in particular, we obtain also that $\not{\Omega}\equiv d\omega+\omega\wedge\omega$, or, equivalently, the following relation (with the Einstein's convention on summations):

$$\not{\Omega}^\alpha_{\mathfrak{B},\beta} = d\omega^\alpha_{\mathfrak{B},\beta}+\omega^\alpha_{\mathfrak{B},\gamma}\wedge\omega^\gamma_{\mathfrak{B},\beta}\,. \tag{5.19}$$

Then, considering $\mathfrak{v}_0\equiv\xi$, we deduce from $\pi(\not{\Omega}(\mathfrak{u},\mathfrak{v})\,\xi)=\pi(\not{\Omega}(\mathfrak{u},\mathfrak{v})\,\mathfrak{v}_0)\equiv\pi(\not{\Omega}^\alpha_{\mathfrak{B},0}(\mathfrak{u},\mathfrak{v})\mathfrak{v}_\alpha)=\not{\Omega}^0_{\mathfrak{B},0}(\mathfrak{u},\mathfrak{v})$ and $\pi((i_\mathfrak{u}\omega)\,\xi)=i_\mathfrak{u}\omega^0_{\mathfrak{B},0}=0$ (*i.e.*, $\pi(\omega\,\xi)=\omega^0_{\mathfrak{B},0}=0$) that $\not{\Omega}^0_{\mathfrak{B},0}=\omega^0_{\mathfrak{B},i}\wedge\omega^i_{\mathfrak{B},0}$, and therefore, we have:

$$\Omega\equiv\not{\Omega}-\not{\Omega}^0_{\mathfrak{B},0}\,\mathbb{1}=d\omega+\omega\wedge\omega-(\omega^0_{\mathfrak{B},i}\wedge\omega^i_{\mathfrak{B},0})\,\mathbb{1}\,, \tag{5.20}$$

where $\omega^i_0\equiv d\kappa^i$. Hence, in particular, we have $\Omega^0_{\mathfrak{B},0}=0$ because $\omega^0_{\mathfrak{B},0}=0$, and, more generally, $Tr(\Omega)=(n+1)\,(\omega^i_{\mathfrak{B},0}\wedge\omega^0_{\mathfrak{B},i})$.

Now, before going further, we note the following.

Lemma 5. *Let $\mathfrak{u}$ and $\mathfrak{v}$ be any vector fields in $\chi(\mathbb{R}P^n)$ and $\underline{\mathfrak{u}}\equiv\mathfrak{q}_0(\mathfrak{u})$ and $\underline{\mathfrak{v}}\equiv\mathfrak{q}_0(\mathfrak{v})$ their corresponding horizontal parts; then, we have that their Lie bracket $[\underline{\mathfrak{u}},\underline{\mathfrak{v}}]$ is horizontal, i.e., $\pi([\underline{\mathfrak{u}},\underline{\mathfrak{v}}])=0$.*

[35] Curvature is related to the restricted holonomy group as well-known [AS53, Ambrose-Singer theorem]; But also, and mainly (!): Proposition 3, in [Ehr47a]. Considering integration along an infinitesimal loop $\delta\ell\subset\mathbb{R}P^n$ bounding an infinitesimal surface (parallelogram) δS oriented along the horizontal normalized (with respect to the canonical Euclidean metric) 2-vector $\underline{\mathfrak{u}}\times\underline{\mathfrak{v}}$, then, the infinitesimal horizontal variation $\underline{\delta\mathfrak{w}}$ of any horizontal vector field $\underline{\mathfrak{w}}$ parallel transported along $\delta\ell$ is $\underline{\delta\mathfrak{w}}\simeq\Big(\underline{\Omega(\underline{\mathfrak{u}},\underline{\mathfrak{v}})\underline{\mathfrak{w}}}\Big)\delta S$. Then, if $Tr(\Omega)=0$, it means that the Euclidean norm of $\underline{\mathfrak{w}}$ is not changed when carried out along $\delta\ell$. Identifying horizontal vectors with points in $\mathbb{R}P^{n+1}\subset\mathbb{R}P^n$, this invariance of the horizontal norm of $\underline{\mathfrak{w}}$ corresponds to a particular local conformal transformation (3.8) of $\mathbb{R}P^n$ called (local) *Möbius transformation* of the inhomogeneous coordinates and volume preserving, *i.e.*, the restricted holonomy group is then $PSL(n,\mathbb{R})$ locally isomorphic to the so-called *Möbius group* $PGL(2,\mathbb{C})$ acting on $\mathbb{R}P^n$.

Proof. Indeed, since π is integrable then there exists a 1-form ρ such that $d\pi = \rho\wedge\pi$. Therefore, because $\pi\circ\mathfrak{q}_0 = 0$, we have that $d\pi(\underline{\mathfrak{u}},\underline{\mathfrak{v}}) = i_{\underline{\mathfrak{u}}}d(i_{\underline{\mathfrak{v}}}\pi) - i_{\underline{\mathfrak{v}}}d(i_{\underline{\mathfrak{u}}}\pi) - \pi([\underline{\mathfrak{u}},\underline{\mathfrak{v}}]) = \pi([\underline{\mathfrak{v}},\underline{\mathfrak{u}}])$, and then, we obtain that $\pi([\underline{\mathfrak{v}},\underline{\mathfrak{u}}]) = (\rho\wedge\pi)(\underline{\mathfrak{v}},\underline{\mathfrak{u}}) = \rho(\underline{\mathfrak{v}})\,\pi(\underline{\mathfrak{u}}) - \rho(\underline{\mathfrak{u}})\,\pi(\underline{\mathfrak{v}}) = 0$. □

Besides, from the same argument, we can also notice that we have necessarily $[\xi,\underline{\mathfrak{u}}] = \rho(\underline{\mathfrak{u}})\,\xi + \underline{\mathfrak{w}}$. But, because the Lie derivative $\mathcal{L}_{\underline{\mathfrak{v}}}$ of $\underline{\mathfrak{v}}$ is a locally transitive (surjective) map[36] on the space of (horizontal) smooth vector fields (we have $\mathcal{L}_{\underline{\mathfrak{v}}}\underline{\mathfrak{u}} \equiv [\underline{\mathfrak{v}},\underline{\mathfrak{u}}]$), we can always find a horizontal vector field $\underline{\mathfrak{v}}$ such that $[\xi+\underline{\mathfrak{v}},\underline{\mathfrak{u}}] \equiv \alpha_{\underline{\mathfrak{v}}}(\underline{\mathfrak{u}})\,(\xi+\underline{\mathfrak{v}})$ where $\alpha_{\underline{\mathfrak{v}}}\in T^*\mathbb{R}P^n$. Hence, we have a degree of freedom in the choice of ξ such that $\pi(\xi)=1$, and therefore, we can redefine ξ such that for any horizontal vector field $\underline{\mathfrak{u}}$ then $[\xi,\underline{\mathfrak{u}}]$ is vertical. Thus, if ξ is such that $[\xi,\underline{\mathfrak{u}}] = \rho(\underline{\mathfrak{u}})\,\xi$ and if we define the finite Lie algebras L_0 and L_1 such that $L_1 = \{\xi\}$ and $L_0 = \{\underline{\mathfrak{u}}_1,\ldots,\underline{\mathfrak{u}}_n\}$ is a set of n horizontal vectors, then we have a semidirect sum $L_1 \rtimes L_0$ (which is a Levi-Malčev decomposition only if L_0 is a semi-simple Lie algebra).

B. The torsion-free and normal projective connections on $\mathbb{R}P^n$

Theorem 6. *Let* $\mathfrak{u}$ *and* $\mathfrak{v}$ *be any vector fields in* $\chi(\mathbb{R}P^n)$*, and let* $\widetilde{\mathfrak{T}}$ *be the* torsion tensor *defined from the projective Cartan derivative* ∇ *(such that* $\nabla_{\mathfrak{u}}\mathfrak{v} = \not\nabla_{\mathfrak{u}}\mathfrak{v} - \pi(\not\nabla_{\mathfrak{u}}\xi)\mathfrak{v}$*) associated with the projective Cartan connection* ω*, i.e.,* $\widetilde{\mathfrak{T}}(\mathfrak{u},\mathfrak{v}) \equiv \nabla_{\mathfrak{u}}\mathfrak{v} - \nabla_{\mathfrak{v}}\mathfrak{u} - [\mathfrak{u},\mathfrak{v}]$*; then, we have*

$$\Omega(\mathfrak{u},\mathfrak{v})\xi = \widetilde{\mathfrak{T}}(\underline{\mathfrak{u}},\underline{\mathfrak{v}}) - \pi\left(\widetilde{\mathfrak{T}}(\underline{\mathfrak{u}},\underline{\mathfrak{v}})\right)\xi\,. \tag{5.21}$$

Proof. First, we must note that $\not\Omega(\mathfrak{u},\mathfrak{v}) = \not\Omega(\underline{\mathfrak{u}},\underline{\mathfrak{v}})$ because $\not\Omega$ can be defined from $\not\omega$ which is a projective connection, and thus, such that $i_{\mathfrak{u}}\not\omega \equiv i_{\underline{\mathfrak{u}}}\not\omega$ since $i_\xi\not\omega = 0$. The same property holds for $\widetilde{\Omega}$. Then, from the relation (5.9) and the relation 2 of theorem 4 (p. 59), we deduce that

$$\begin{aligned}\not\Omega(\mathfrak{u},\mathfrak{v})\xi &= \nabla_{\underline{\mathfrak{u}}}\nabla_{\underline{\mathfrak{v}}}\xi - \nabla_{\underline{\mathfrak{v}}}\nabla_{\underline{\mathfrak{u}}}\xi - \nabla_{[\underline{\mathfrak{u}},\underline{\mathfrak{v}}]}\xi + d\not\chi(\underline{\mathfrak{u}},\underline{\mathfrak{v}})\xi\\ &= \nabla_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}} - \nabla_{\underline{\mathfrak{v}}}\underline{\mathfrak{u}} - [\underline{\mathfrak{u}},\underline{\mathfrak{v}}] + d\not\chi(\underline{\mathfrak{u}},\underline{\mathfrak{v}})\xi\\ &= \widetilde{\mathfrak{T}}(\underline{\mathfrak{u}},\underline{\mathfrak{v}}) + d\not\chi(\underline{\mathfrak{u}},\underline{\mathfrak{v}})\xi\,.\end{aligned} \tag{5.22}$$

[36] In codimension 1: the notion of transversal parallelizability if a non-singular closed Pfaff 1-form exists on the whole of the manifold, and in codimension q: if there are q *basic* Pfaff 1-forms [God71, see pp.158–160]. Then, by duality, we obtain infinitesimal transitive local actions of pseudogroups on the foliation.

And then, we deduce also that $\pi\left(\cancel{\Omega}(\mathfrak{u},\mathfrak{v})\xi\right) = \pi\left(\widetilde{\mathfrak{T}}(\underline{\mathfrak{u}},\underline{\mathfrak{v}})\right) + d\cancel{\chi}(\underline{\mathfrak{u}},\underline{\mathfrak{v}})$; hence, the result from (5.11a). □

Then, we have the following definition.

Definition 16. *Let* $\mathfrak{u}$ *and* $\mathfrak{v}$ *be any vector fields in* $\chi(\mathbb{R}P^n)$*. We denote by* $\mathfrak{T}$ *the* projective Cartan torsion tensor *defined by the projective Cartan connection* ω *associated with* ∇ *and such that*

$$\mathfrak{T}(\mathfrak{u},\mathfrak{v}) \equiv \Omega(\mathfrak{u},\mathfrak{v})\xi = \underline{\nabla_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} - \underline{\nabla_{\underline{\mathfrak{v}}}\underline{\mathfrak{u}}} - [\underline{\mathfrak{u}},\underline{\mathfrak{v}}]\,. \tag{5.23}$$

Then, a projective Cartan connection ω *is said* 'torsion-free' *if the covariant projective Cartan derivative it defines is such that* $\mathfrak{T} \equiv 0$*. Also, we can note that* $\mathfrak{p}_0(\mathfrak{T}) = 0$*.*

We have important results from the projective Cartan connection ω. The latter defines 1) two torsion tensors:

- $\widetilde{\mathfrak{T}}$ which is the Riemannian torsion given in Theorem 6 and
- $\mathfrak{T}$ which is the projective torsion given in (5.23),

and 2) two curvature tensors:

- $\widetilde{\Omega}$ which is the Riemannian curvature tensor given in (5.4) or (5.5) and
- Ω which is the projective curvature tensor given in (5.17).

Moreover, we obtained 1) the relation (5.12): $\Omega = \widetilde{\Omega} - \pi\left(\widetilde{\Omega}\,.\,\xi\right)\mathbb{1}$, from the definitions (5.5) and (5.11), and 2) we obtain from (5.23) the relation $\mathfrak{T}(\mathfrak{u},\mathfrak{v}) = \Omega(\mathfrak{u},\mathfrak{v})\,\xi = \widetilde{\Omega}(\mathfrak{u},\mathfrak{v})\,\xi - \pi\left(\widetilde{\Omega}(\mathfrak{u},\mathfrak{v})\,\xi\right)\xi$. And therefore, we deduce that

$$\mathfrak{T}(\mathfrak{u},\mathfrak{v}) = \widetilde{\mathfrak{T}}(\mathfrak{u},\mathfrak{v}) - \pi((\omega\wedge\omega)(\mathfrak{u},\mathfrak{v})\xi)\xi\,. \tag{5.24}$$

But, if $Tr(\omega_{\mathfrak{B}}) = 0$, then, from (5.18), we have

$$\pi(\widetilde{\Omega}\,\xi) = \pi((\omega\wedge\omega)\xi) = -\frac{1}{(n+1)}Tr(\Omega)\,. \tag{5.25}$$

Therefore, we obtain:

Theorem 7. *If ω is a projective Cartan connection, then we have*

$$\widetilde{\Omega} = \Omega - \frac{1}{(n+1)}\, Tr(\Omega)\, \mathbb{1}\,, \tag{5.26a}$$

$$\widetilde{\mathfrak{T}} = \mathfrak{T} - \frac{1}{(n+1)}\, Tr(\Omega)\, \xi\,. \tag{5.26b}$$

This theorem indicates clearly that these projective tensors are equal to their Riemannian counterparts if and only if $Tr(\Omega) = 0$. And thus, if $Tr(\Omega) = 0$ and ω is equivalent to a torsion-free Euclidean connection then, necessarily, the projective torsion of ω is also vanishing.

And also, we conclude the following.

Property 2. *If ω is a projective Cartan connection such that $Tr(\Omega) = 0$, then we have*

$$\Omega = d\omega + \omega \wedge \omega\,. \tag{5.27}$$

Example 2. Then, considering again the previous example in the general case due to É. Cartan, if ω is torsion-free, *i.e.*, $\mathfrak{T} \equiv 0$, then, we have that $\Omega\,.\,\xi \equiv \Omega\,.\,\mathfrak{v}_0 = \sum_{i=1}^{n} \Omega_0^i\, \mathfrak{v}_i = 0$. Hence, the vanishing of the projective Cartan torsion tensor is expressed by the n relations:

$$\Omega_0^i = 0\,, \qquad (i = 1, \ldots, n), \tag{5.28}$$

or, equivalently, from (5.20), the n relations

$$d\kappa^k \wedge \omega_{\mathfrak{B},k}^i = 0 \tag{5.29}$$

which were given by É. Cartan. Moreover, from the defining relations (5.23), we have $\mathfrak{T}(\mathfrak{v}_i, \mathfrak{v}_j) = \underline{\nabla_{\mathfrak{v}_i}\mathfrak{v}_j} - \underline{\nabla_{\mathfrak{v}_j}\mathfrak{v}_i} - [\mathfrak{v}_i, \mathfrak{v}_j] = 0$, *i.e.*, we obtain that

$$\omega_{\mathfrak{B},j}^k(\mathfrak{v}_i) - \omega_{\mathfrak{B},i}^k(\mathfrak{v}_j) = \mathfrak{v}^{*k}([\mathfrak{v}_i, \mathfrak{v}_j]) \tag{5.30}$$

which are, no more no less, the relations satisfied by any torsion-free Euclidean Levi-Civita connection but restricted only, in the present particular case, to the horizontal space. In particular, if $\mathfrak{v}_i \equiv \partial_{\kappa^i}$ is the dual vector of $d\kappa^i$, then, $[\mathfrak{v}_i, \mathfrak{v}_j] = 0$ and if we use the notation

$$\gamma_{j,i}^k \equiv \omega_{\mathfrak{B},j}^k(\mathfrak{v}_i)\,, \tag{5.31}$$

we recognize the very well-known symmetry of the Christoffel symbols γ but in the horizontal space only:

$$\gamma^k_{j,i} = \gamma^k_{i,j}\,. \tag{5.32}$$

Nevertheless, if $[\mathfrak{v}_i, \mathfrak{v}_j] \neq 0$ and if the horizontal space (of dimension $n-1$) is provided with the canonical Euclidean metric η, then, the compatibility condition $\underline{\nabla\eta} = 0$ involves that $\gamma^k_{j,i} = -\gamma^j_{k,i}$. Then, we have: $\gamma^k_{j,i} = \gamma^k_{i,j} + \mathfrak{v}^{*k}([\mathfrak{v}_i, \mathfrak{v}_j]) = -\gamma^i_{k,j} + \mathfrak{v}^{*k}([\mathfrak{v}_i, \mathfrak{v}_j]) = -\gamma^i_{j,k} - \mathfrak{v}^{*i}([\mathfrak{v}_j, \mathfrak{v}_k]) + \mathfrak{v}^{*k}([\mathfrak{v}_i, \mathfrak{v}_j]) = \gamma^j_{i,k} - \mathfrak{v}^{*i}([\mathfrak{v}_j, \mathfrak{v}_k]) + \mathfrak{v}^{*k}([\mathfrak{v}_i, \mathfrak{v}_j]) = \gamma^j_{k,i} + \mathfrak{v}^{*j}([\mathfrak{v}_k, \mathfrak{v}_i]) - \mathfrak{v}^{*i}([\mathfrak{v}_j, \mathfrak{v}_k]) + \mathfrak{v}^{*k}([\mathfrak{v}_i, \mathfrak{v}_j]) = -\gamma^k_{j,i} + \mathfrak{v}^{*j}([\mathfrak{v}_k, \mathfrak{v}_i]) - \mathfrak{v}^{*i}([\mathfrak{v}_j, \mathfrak{v}_k]) + \mathfrak{v}^{*k}([\mathfrak{v}_i, \mathfrak{v}_j])$. Hence, we deduce the well-known relation:

$$\gamma^k_{j,i} = \frac{1}{2}\left(\mathfrak{v}^{*j}([\mathfrak{v}_k, \mathfrak{v}_i]) - \mathfrak{v}^{*i}([\mathfrak{v}_j, \mathfrak{v}_k]) + \mathfrak{v}^{*k}([\mathfrak{v}_i, \mathfrak{v}_j])\right). \tag{5.33}$$

Furthermore, we have the following important relations pointing out constraints on the projective Cartan curvature Ω:

Property 3. *Let* $\mathfrak{u}$, $\mathfrak{v}$ *and* $\mathfrak{w}$ *be any three vector fields over* $\mathbb{R}P^n$. *Let* Ω *be the projective Cartan curvature defined from the relation* (5.11b) *by the* torsion-free *projective Cartan connection* ω*; then,* Ω *satisfies the* $n+1$ *following relations:*

$$\underline{(\Omega\wedge\omega)\xi} = 0\,, \tag{5.34a}$$

or, equivalently,

$$\mathfrak{q}_0\Big(\big(\Omega(\mathfrak{u},\mathfrak{v})\,\omega(\mathfrak{w}) + \Omega(\mathfrak{v},\mathfrak{w})\,\omega(\mathfrak{u}) + \Omega(\mathfrak{w},\mathfrak{u})\,\omega(\mathfrak{v})\big)\xi\Big) = 0\,. \tag{5.34b}$$

Moreover, if $Tr(\Omega) = 0$*, then, we have* $(\Omega\wedge\omega)\xi = 0$ *and* $\pi((\omega\wedge\omega)\,.\,\xi) = 0\,.$

Proof. Indeed, From $\Omega\,.\,\xi = 0$ we deduce that $(d\Omega).\,\xi = 0$ and $\not\Omega\,.\,\xi = \not\Omega^0_{\mathfrak{B},0}\,\xi$ where $\not\Omega^0_{\mathfrak{B},0} \equiv \pi(\not\Omega\,.\,\xi)$, and therefore, $d\not\Omega^0_{\mathfrak{B},0}\,\xi = (d\not\Omega)\,.\,\xi\,.$ Then, from the relation $d\not\Omega = \not\Omega\wedge\not\omega - \not\omega\wedge\not\Omega$, we deduce that $d\not\Omega^0_{\mathfrak{B},0}\,\xi = \not\Omega\wedge\not\omega\,.\,\xi - \not\omega\wedge\not\Omega^0_{\mathfrak{B},0}\,\xi = \left(\not\Omega - \not\Omega^0_{\mathfrak{B},0}\mathbb{1}\right)\wedge\not\omega\,.\,\xi = \Omega\wedge\not\omega\,.\,\xi = \Omega\wedge(\omega + \not\omega^0_{\mathfrak{B},0}\mathbb{1})\,.\,\xi$ where $\not\omega^0_{\mathfrak{B},0} \equiv \pi(\not\omega\,.\,\xi)\,.$ Then, we obtain also that $d\not\Omega^0_{\mathfrak{B},0}\,\xi = \Omega\wedge(\omega + \not\omega^0_{\mathfrak{B},0}\mathbb{1})\,.\,\xi = \Omega\wedge\omega\,.\,\xi - \not\omega^0_{\mathfrak{B},0}\wedge\Omega\,.\,\xi = \Omega\wedge\omega\,.\,\xi\,$; hence the first result.

Now, if $Tr(\Omega) = 0$, then $\Omega = \widetilde{\Omega}$ and therefore, $d\Omega = \Omega\wedge\omega - \omega\wedge\Omega$. Thus, $d\Omega\,.\,\xi = 0 = \Omega\wedge\omega\,.\,\xi - \omega\wedge\Omega\,.\,\xi = \Omega\wedge\omega\,.\,\xi\,.$

Lastly, from $\Omega = d\omega + \omega\wedge\omega$, $\pi(d\omega\,.\,\xi) = 0$ and $\Omega\,.\,\xi = 0$ we deduce the last relation. □

Example 3. Following Example 2, we have, actually, n relations rather than $n+1$ because $\Omega^0_{\mathfrak{B},0} = 0$ from (5.20). Now, let $\mathfrak{u}$ be such that $\mathfrak{u} \equiv \mathfrak{u}^\alpha\, \mathfrak{v}_\alpha$. Then, first, we have $\omega\, \mathfrak{u} = \mathfrak{u}^\alpha\, \omega^\beta_{\mathfrak{B},\alpha}\, \mathfrak{v}_\beta$. And second, $(\Omega \wedge \omega)\, \mathfrak{u} = \mathfrak{u}^\alpha\, (\Omega\, \mathfrak{v}_\beta) \wedge \omega^\beta_{\mathfrak{B},\alpha} = \mathfrak{u}^\alpha\, (\Omega^\gamma_{\mathfrak{B},\beta}\, \mathfrak{v}_\gamma) \wedge \omega^\beta_{\mathfrak{B},\alpha} = \mathfrak{u}^\alpha\, (\Omega^\gamma_{\mathfrak{B},\beta} \wedge \omega^\beta_{\mathfrak{B},\alpha})\, \mathfrak{v}_\gamma$. Then, with $\mathfrak{u} \equiv \xi \equiv \mathfrak{v}_0$, we obtain that $(\Omega \wedge \omega)\, \xi = (\Omega^\gamma_{\mathfrak{B},\beta} \wedge \omega^\beta_{\mathfrak{B},0})\, \mathfrak{v}_\gamma$. Moreover, we have $\omega^0_{\mathfrak{B},0} = 0$. Therefore, we deduce that $(\Omega \wedge \omega)\, \xi = (\Omega^\gamma_{\mathfrak{B},i} \wedge \omega^i_{\mathfrak{B},0})\, \mathfrak{v}_\gamma$ where, again, $\omega^i_{\mathfrak{B},0} \equiv d\kappa^i$. Therefore, with $\Omega^i_{\mathfrak{B},0} = 0$, $\Omega^0_{\mathfrak{B},0} = 0$, $\Omega^\alpha_{\mathfrak{B},\beta} = \Omega^\alpha_{\mathfrak{B},\beta,\mu\nu}\, \omega^\mu_{\mathfrak{B},0} \wedge \omega^\nu_{\mathfrak{B},0}$ and because Ω is horizontal, then, from (5.34) we obtain that

$$\Omega^\alpha_{\mathfrak{B},0,\mu\nu} = 0\,, \tag{5.35a}$$

$$\Omega^\alpha_{\mathfrak{B},\beta,0\nu} = 0\,, \tag{5.35b}$$

$$\Omega^h_{\mathfrak{B},i,jk} + \Omega^h_{\mathfrak{B},j,ki} + \Omega^h_{\mathfrak{B},k,ij} = 0\,. \tag{5.35c}$$

Additionally, if $Tr(\Omega) = (n+1)(\omega^0_{\mathfrak{B},i} \wedge \omega^i_{\mathfrak{B},0}) = 0$ (and consequently, any holonomy action is a local conformal transformation (3.8) which is an element of the Möbius group (see footnote 35)) we obtain also

$$\Omega^0_{\mathfrak{B},i,jk} + \Omega^0_{\mathfrak{B},j,ki} + \Omega^0_{\mathfrak{B},k,ij} = 0\,. \tag{5.36}$$

And, if we set $\omega^0_{\mathfrak{B},i} \equiv \gamma^0_{i,j}\, \omega^j_{\mathfrak{B},0}$, then, we deduce also that

$$\gamma^0_{i,j} = \gamma^0_{j,i} \tag{5.37}$$

in addition to the relations (5.32).

Lastly, we deduce the expression usually presented in the "*literature*" to define the projective equivalence between projective derivatives:

Property 4. *Let* $\widetilde{\nabla}$ *and* ∇ *be two torsion-free projective covariant derivatives such that* $\widetilde{\nabla} \sim_P \nabla$ *and* $\mathfrak{p}_0(\widetilde{\nabla}\xi) = \mathfrak{p}_0(\nabla\xi) = 0$*; then we have that*

$$\underline{\widetilde{\nabla}_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} = \underline{\nabla_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} + \theta(\underline{\mathfrak{u}})\underline{\mathfrak{v}} + \theta(\underline{\mathfrak{v}})\underline{\mathfrak{u}}\,, \tag{5.38}$$

where $\theta \in T^*\mathbb{R}P^n$ *is a scalar 1-form.*

Proof. Assuming $\widetilde{\mathfrak{T}} \equiv 0$, $\mathfrak{T} \equiv 0$, then, from the corollary 2, we must have $\mathfrak{r}_1 = 0$ to have $\mathfrak{p}_0(\widetilde{\mathfrak{T}}) = \mathfrak{p}_0(\mathfrak{T}) = 0$, *i.e.*, $\widetilde{\Omega}\,.\,\xi = \underline{\widetilde{\Omega}\,.\,\xi}$ and $\Omega\,.\,\xi = \underline{\Omega\,.\,\xi}$ where $\widetilde{\Omega}$ and Ω are defined from the two projectively equivalent projective connections $\tilde{\omega}$ and ω by the formula (5.11b). Moreover, the condition $\mathfrak{p}_0(\widetilde{\nabla}\xi) = \mathfrak{p}_0(\nabla\xi) = 0$ must be set since it is assumed in the proof of Corollary 2. Equivalently, the conditions $\tilde{\omega}^0_{\mathfrak{B},0} = \omega^0_{\mathfrak{B},0} = 0$ must be satisfied. Then, necessarily, we must take $\mathfrak{A}$ such that $\mathfrak{A} \equiv \mathfrak{s} = \mathfrak{r}_n\,\mathbb{1}_n + \tilde{\mathfrak{s}}$. But, also, from the relation (5.16), if the two torsions are equal then necessarily, the horizontal map $\underline{\tilde{\mathfrak{s}}}$ such that $\underline{\tilde{\mathfrak{s}}}\,\underline{\mathfrak{u}} \equiv \tilde{\mathfrak{s}}\,\underline{\mathfrak{u}}$ must vanish.

Then, we obtain $\widetilde{\nabla}_{\mathfrak{u}}\mathfrak{v} = \nabla_{\mathfrak{u}}\mathfrak{v} + (i_{\mathfrak{u}}\mathfrak{s})\mathfrak{v}$ as in (3.37) and also the following relations:

$$\begin{aligned}
\underline{\widetilde{\nabla}_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} &= \frac{1}{2}\,\underline{\widetilde{\nabla}_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} + \frac{1}{2}\,\underline{\widetilde{\nabla}_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} \\
&= \frac{1}{2}\,\underline{\widetilde{\nabla}_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} + \frac{1}{2}\,\underline{\widetilde{\nabla}_{\underline{\mathfrak{v}}}\underline{\mathfrak{u}}} + \frac{1}{2}\,[\underline{\mathfrak{u}},\underline{\mathfrak{v}}] \\
&= \frac{1}{2}\,\underline{\nabla_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} + \frac{1}{2}\,\underline{\nabla_{\underline{\mathfrak{v}}}\underline{\mathfrak{u}}} + \frac{1}{2}\,(i_{\underline{\mathfrak{u}}}\mathfrak{s})\underline{\mathfrak{v}} + \frac{1}{2}\,(i_{\underline{\mathfrak{v}}}\mathfrak{s})\underline{\mathfrak{u}} + \frac{1}{2}\,[\underline{\mathfrak{u}},\underline{\mathfrak{v}}] \\
&= \underline{\nabla_{\underline{\mathfrak{u}}}\underline{\mathfrak{v}}} + \frac{1}{2}\,(i_{\underline{\mathfrak{u}}}\mathfrak{s})\underline{\mathfrak{v}} + \frac{1}{2}\,(i_{\underline{\mathfrak{v}}}\mathfrak{s})\underline{\mathfrak{u}}\,.
\end{aligned} \tag{5.39}$$

Then, $\theta(\underline{\mathfrak{u}})\underline{\mathfrak{v}} \equiv (i_{\underline{\mathfrak{u}}}\mathfrak{s})\underline{\mathfrak{v}}/2 = (i_{\underline{\mathfrak{u}}}\mathfrak{r}_1)\underline{\mathfrak{v}}/2 + (i_{\underline{\mathfrak{u}}}\tilde{\mathfrak{s}})\underline{\mathfrak{v}}/2 = (i_{\underline{\mathfrak{u}}}\mathfrak{r}_1)\underline{\mathfrak{v}}/2$ because $\underline{\tilde{\mathfrak{s}}} = 0$, and therefore, also, $\theta \in T^*\mathbb{R}P^n$. □

Now, we have the following definition for the so-called *normal* projective Cartan connections:

Definition 17. *Let ω be a torsion-free projective Cartan connection (Definition 16) and Ω its corresponding projective Cartan curvature (formula* (5.11b)*) such that $Tr(\Omega) = 0$, and then (formula* (5.27)*), we have $\Omega = d\omega + \omega \wedge \omega$. Moreover, if we denote by Ωic the Ricci tensor defined exactly as in the Riemannian case, i.e., $i_{\mathfrak{u}}\Omega ic \equiv Tr(\Upsilon\,.\,\mathfrak{u})$ for any vector field $\mathfrak{u} \in \chi(\mathbb{R}P^n)$ where Υ is the map such that $\Upsilon : \mathfrak{v} \in \chi(\mathbb{R}P^n) \longrightarrow -i_{\mathfrak{v}}\Omega \in \Gamma_n((\mathbb{R}\oplus stab([\xi])) \otimes T^*\mathbb{R}P^n)$ and $\Upsilon\,.\,\mathfrak{u} \in \Gamma_n(\chi(\mathbb{R}P^n) \otimes T^*\mathbb{R}P^n)$, then, ω is said to be a* normal *projective Cartan connection if, additionally, $\Omega ic = 0$.*

Example 4. Continuing the previous examples, ω is *normal* if we add to the relations (5.32), (5.35) and (5.37) the $n(n+1)/2$ relations:

$$\Omega ic_{\mathfrak{B},\alpha\beta} \equiv \Omega^{\mu}_{\mathfrak{B},\alpha,\mu\beta} = \Omega^{k}_{\mathfrak{B},\alpha,k\beta} = 0\,. \tag{5.40}$$

Remark 13. *The deep meaning of a* normal *projective Cartan connection is linked to the projective geodesics on $\mathbb{R}P^n$. More precisely, É. Cartan proved that given a particular system of differential equations of which the integral curves define the whole set of projective geodesics, then, this system of differential equation defines a particular set of torsion-free projective Cartan connections. And then, he shown that among this set of projective connections, there exists one and only one connection which is normal.*

Actually, the conditions $\Omega ic = 0$ and $Tr(\Omega) = 0$ set by É. Cartan are chosen explicitly to give a univocal definition of the so-called soldering forms — which are the 1-forms ω_i^0 in the example 4 just above — from the 1-forms ω_j^i constituting the horizontal part of the projective (affine) connections ω; the latter being themselves completely and univocally determined from the given set of projective geodesics viewed as integral curves of a given system of differential equations validated as projective geodesic equations from their "shapes."

Thus, we have the important following consequence: if we choose explicitly a given set of 1-forms ω_i^0 and if we have at disposal a system of differential equations satisfying the criteria to be projective geodesic equations compatible with the previous choice, then, ω is univocally and directly defined as a normal projective connection.

Actually, É. Cartan chose the soldering forms ω_i^0 as to be the 1-forms decomposing a given horizontal metric so that the horizontal part (ω_j^i) of ω is necessarily in the set of Lie algebra-valued 1-forms preserving this given metric, i.e., (ω_j^i) is a pull-back of the Maurer-Cartan 1-form of the Lie algebra preserving the quadratic form defined by the metric.

In the present context of projective spacetime geometry yielded by relativistic positioning systems, and because the light-like conformal geodesics and the timelike-like projective geodesics for free falling particles are at the ground of the spacetime geometry — as it has been shown, for instance, in the various causal axiomatics at disposal for the spacetime (see [KP67, Car71, Woo73, HKM76, Mal77] for instance), the normal projective Cartan connections and the conformal metric g, both together, appear to be perfectly adequate and sufficient for the spacetime geometry description thought not complete. Indeed, because of the projective (scaling) indetermination coming from the physical arbitrariness in the choice of the "coordinating time stamps τ_α," we can only expect to know the geometry of $\mathcal{M}_{RPS}$ up to a certain "scaling coor-

dinate" defined somehow from the τ_α's, actually, κ^1. But, nevertheless, we shall also indicate a physical complementary process provided by any relativistic positioning system giving a true complete geometrical description for the true spacetime $\mathcal{M}\,(\neq \mathcal{M}_{RPS})$, and thus, allowing to access to $\mathcal{M}$.

Remark 14. *Besides, É. Cartan proved the following theorem [Car24b, Chap. VI. §14, p. 226 and VI. §15, p. 227] (see also [Lev96, Tho25, Eas08, EM08, LC09]).*

> **Theorem 8.** *Let M be a manifold endowed with a Euclidean metric and a Euclidean compatible connection providing M with a Riemaniann structure. Then, M is locally projectively flat manifold if and only if its Riemaniann curvature tensor is constant.*

The meaning of a projectively flat manifold is that the Euclidean geodesics on M (i.e., the geodesics on M with respect its Euclidean connection) are also projective geodesics with respect to a projective flat connection defined on M of which the horizontal part is the given Euclidean connection ω. Indeed, the latter can be considered as the horizontal restriction $\omega \equiv \underline{\varpi}$ to T^*M of a normal *projective connection ϖ. Then, considering implicitly that M is a leaf of a codimension one foliation, and denoting by Π the projective curvature tensor defined by ϖ and Ω the Riemann curvature tensor defined by ω, Cartan shown that the horizontal part $\underline{\Pi}$ of Π can be written in the form $\underline{\Pi} = \Omega + \Sigma$ where Σ is a particular curvature tensor. Then, from the property ${}^t\Omega = -\Omega$, the conditions (5.34) satisfied by Π and the vanishing of the projective torsion, he deduced certain properties on Σ which constraint the Riemaniann curvature tensor Ω to be constant when $\underline{\Pi} = 0$.*

VI. THE NORMAL PROJECTIVE CURVATURE OF THE THREE-DIMENSIONAL SUBMANIFOLDS OF $\mathcal{M}_{RPS}$ MODELED ON $\mathbb{R}P^3$

First, we list the set of properties the projective Cartan connection and the projective curvature satisfy. We have the following for the projective Cartan connection:

- $\omega^1_{\mathfrak{B},1} = 0,$ (6.1a)
- $\omega^i_{\mathfrak{B},1} = \pi = d\kappa^i,$ (6.1b)
- $Tr(\omega_{\mathfrak{B}}) = 0,$ (6.1c)
- $i_\xi \omega_{\mathfrak{B}} = 0,$ (6.1d)
- $i_\xi \omega_{\mathfrak{B}''} = -\dfrac{1}{2}\,\mathbb{1},$ (6.1e)

and for the projective curvature:

- $\Omega^1_{\mathfrak{B},1} = 0,$ (6.2a)
- $i_\xi \Omega = 0.$ (6.2b)

Now, we assume that the projective Cartan connection is *normal*, and therefore, we have, in addition to the previous relations, the following:

- $Tr(\Omega) = -5\,\pi\Big((\omega \wedge \omega)\,.\,\xi\Big) = 0,$ (6.3a)
- $\mathfrak{T} \equiv \Omega\,.\,\xi = 0,$ (6.3b)
- $\Omega ic = 0.$ (6.3c)

And therefore, we deduce that

- $\Omega = d\omega + \omega \wedge \omega,$ (6.4a)
- $(\Omega \wedge \omega)\,.\,\xi = 0\,,$ (6.4b)
- $\Omega^\alpha_{\mathfrak{B},1} = 0.$ (6.4c)

Now, the relations (4.80n) and (4.80o) are equivalent to $\nabla g = 2\,\theta \otimes g$ where we denote by ∇ the covariant derivative associated with the projective Cartan connection ω. Also, denoting by

$\widetilde{\nabla}$ the covariant derivative associated with the metric connection Γ compatible with the metric g of signature $(+,-,-,-)$, and thus, such that $\widetilde{\nabla} g=0$, then, from the relations $i_{\mathfrak{w}}d(g(\mathfrak{u},\mathfrak{v}))=g(\widetilde{\nabla}_{\mathfrak{w}}\mathfrak{u},\mathfrak{v})+g(\mathfrak{u},\widetilde{\nabla}_{\mathfrak{w}}\mathfrak{v})$ and $i_{\mathfrak{w}}d(g(\mathfrak{u},\mathfrak{v}))=(\nabla_{\mathfrak{w}}g)(\nabla_{\mathfrak{w}}\mathfrak{u},\mathfrak{v})+g(\nabla_{\mathfrak{w}}\mathfrak{u},\mathfrak{v})+g(\mathfrak{u},\nabla_{\mathfrak{w}}\mathfrak{v})$, we deduce that for all vector fields $\mathfrak{u}$ and $\mathfrak{v}$ we have

$$\widetilde{\nabla}_{\mathfrak{u}}\,\mathfrak{v}=\nabla_{\mathfrak{u}}\,\mathfrak{v}+S_\Theta(\mathfrak{u},\mathfrak{v})\,, \tag{6.5}$$

where S_Θ is, by definition [Kul70, p.317, noting that Kulkarni gives a definition of the Riemann tensor differing by a sign from the definition used presently], such that (see also Appendix J with $\varphi\equiv\Theta$ and $G\equiv\Psi$) such that

$$\theta=d\Theta=\frac{1}{2}\,\pi, \tag{6.6a}$$

$$S_\Theta(\mathfrak{u},\mathfrak{v})=(i_{\mathfrak{u}}\theta)\,\mathfrak{v}+(i_{\mathfrak{v}}\theta)\,\mathfrak{u}-g(\mathfrak{u},\mathfrak{v})\,\Psi\,, \tag{6.6b}$$

$$\Psi\equiv grad(\Theta)\,, \tag{6.6c}$$

$$g(\mathfrak{u},\Psi)\equiv i_{\mathfrak{u}}\theta\,. \tag{6.6d}$$

We must note that Γ may have a nonvanishing torsion even if the projective torsion is vanishing. The Christoffel symbols are given in Appendix K.

Also, from (6.6d), we deduce in the basis $\mathfrak{B}$ that $g(\mathfrak{v}_\alpha,\Psi)=i_{\mathfrak{v}_\alpha}\theta=\frac{1}{2}\,i_{\mathfrak{v}_\alpha}d\kappa^1=\frac{1}{2}\,i_{\mathfrak{v}_\alpha}\pi=\frac{1}{2}\,\delta^1_\alpha$. And thus, we obtain that

$$\Psi\equiv\frac{1}{2}\,\xi\,,\qquad \|\Psi\|^2=g(\Psi,\Psi)=\frac{1}{4}\,,\qquad g(\xi,\xi)=1\,. \tag{6.7}$$

Hence, ξ is a normalized time-like vector with respect to the metric g. Besides, from the results in the appendix J with $\varphi\equiv\Theta$, we obtain the general relation

$$R(\mathfrak{u},\mathfrak{v})\mathfrak{w}=\Omega(\mathfrak{u},\mathfrak{v})\mathfrak{w}-T(\mathfrak{u},\mathfrak{v})\mathfrak{w}\,, \tag{6.8}$$

where

$$T(\mathfrak{u},\mathfrak{v})\mathfrak{w} = \Big(Q(\mathfrak{v},\mathfrak{w}) + \frac{1}{4}\,g(\mathfrak{v},\mathfrak{w})\Big)\,\mathfrak{u} - \Big(Q(\mathfrak{u},\mathfrak{w}) + \frac{1}{4}\,g(\mathfrak{u},\mathfrak{w})\Big)\,\mathfrak{v} \\ + g(\mathfrak{v},\mathfrak{w})\,Q_0(\mathfrak{u}) - g(\mathfrak{u},\mathfrak{w})\,Q_0(\mathfrak{v})\,, \tag{6.9a}$$

$$Q(\mathfrak{u},\mathfrak{v}) = hess_\Theta(\mathfrak{u},\mathfrak{v}) - \theta(\mathfrak{u})\,\theta(\mathfrak{v})\,, \tag{6.9b}$$

$$hess_\Theta(\mathfrak{u},\mathfrak{v}) = \frac{1}{2}\,g(\mathfrak{u},\nabla_{\mathfrak{v}}\xi)\,, \tag{6.9c}$$

$$Q_0(\mathfrak{u}) \equiv \frac{1}{2}\,(\nabla_{\mathfrak{u}}\xi - \theta(\mathfrak{u})\xi)\,. \tag{6.9d}$$

Then, from the condition 2 in Theorem 4, *i.e.*, $\nabla_{\mathfrak{u}}\xi = \mathfrak{q}_0(\mathfrak{u})$ and $\theta \equiv \frac{1}{2}\,\pi$, the relation

$$Q_0(\mathfrak{u}) \equiv \frac{1}{2}\Big(\mathfrak{u} - \frac{3}{2}\,\pi(\mathfrak{u})\,\xi\Big) \tag{6.10}$$

holds. In addition, we have also , $hess_\Theta(\mathfrak{u},\mathfrak{v}) = \frac{1}{2}\,g(\mathfrak{u},\mathfrak{q}_0(\mathfrak{v})) = \frac{1}{2}\,g(\mathfrak{q}_0(\mathfrak{u}),\mathfrak{q}_0(\mathfrak{v}))$, and thus, the Hessian is defined by the relation:

$$hess_\Theta(\mathfrak{u},\mathfrak{v}) = \frac{1}{2}\,g(\underline{\mathfrak{u}},\underline{\mathfrak{v}})\,. \tag{6.11}$$

Consequently, we deduce that $Q(\mathfrak{u},\mathfrak{v}) = hess_\Theta(\mathfrak{u},\mathfrak{v}) - \theta(\mathfrak{u})\,\theta(\mathfrak{v}) = \frac{1}{2}\,g(\underline{\mathfrak{u}},\underline{\mathfrak{v}}) - \theta(\mathfrak{u})\,\theta(\mathfrak{v}) = \frac{1}{2}\,g(\underline{\mathfrak{u}},\underline{\mathfrak{v}}) - \frac{1}{4}\,\pi(\mathfrak{u})\,\pi(\mathfrak{v})$, and then, we have

$$Q(\mathfrak{u},\mathfrak{v}) = \frac{1}{2}\Big(g(\underline{\mathfrak{u}},\underline{\mathfrak{v}}) - \frac{1}{2}\,\pi(\mathfrak{u})\,\pi(\mathfrak{v})\Big)\,. \tag{6.12a}$$

And, as a result, we deduce also that

$$Q(\mathfrak{u},\mathfrak{v}) + \frac{1}{4}\,g(\mathfrak{v},\mathfrak{w}) = \frac{3}{4}\,g(\underline{\mathfrak{u}},\underline{\mathfrak{v}})\,. \tag{6.12b}$$

From these relations, we obtain the complete expression for the tensor T:

$$T(\mathfrak{u},\mathfrak{v})\,\mathfrak{w} = \frac{1}{4}\,\{5\,[g(\mathfrak{w},\mathfrak{v})\,\mathfrak{u} - g(\mathfrak{w},\mathfrak{u})\,\mathfrak{v}] + 3\,\pi(\mathfrak{w})\,[\pi(\mathfrak{u})\,\mathfrak{v} - \pi(\mathfrak{v})\,\mathfrak{u}] \\ +3\,[\pi(\mathfrak{v})\,g(\mathfrak{u},\mathfrak{w}) - \pi(\mathfrak{u})\,g(\mathfrak{v},\mathfrak{w})]\,\xi\}\,. \tag{6.13}$$

In particular, if $\mathfrak{u} \equiv \xi$, then T is such that

$$T(\xi,\mathfrak{v})\,\mathfrak{w} = \frac{1}{2}\,(g(\mathfrak{v},\mathfrak{w})\,\xi - \pi(\mathfrak{w})\,\mathfrak{v}))\,. \tag{6.14}$$

Consequently, $i_\xi R \neq 0$ in full generality. Moreover, we have also the relation (see Appendix J with $\varphi \equiv \Theta$ again):

$$Tr(R(\mathfrak{u},\mathfrak{v})) = Tr(\Omega(\mathfrak{u},\mathfrak{v})) + \theta(\widetilde{\mathfrak{T}}(\mathfrak{u},\mathfrak{v})) = \theta(\widetilde{\mathfrak{T}}(\mathfrak{u},\mathfrak{v}))\,, \tag{6.15}$$

where $\widetilde{\mathfrak{T}}$ is the Riemannian torsion tensor defined by the connection ω. Additionally, because the Riemannian torsion tensor $\widetilde{\mathfrak{T}}$ equals the projective torsion tensor $\mathfrak{T}\,(=0)$ defined by ω since $Tr(\Omega) = 0$, the relation

$$Tr(R) = 0 \tag{6.16}$$

holds. Moreover, we have (see Appendix J with $n = 4$) $Sc = e^{-2\Theta}\left(\Omega sc + 12\|\varPsi\|^2 + 6\,Tr(Q_0)\right)$, where Ωsc is the scalar curvature defined by Ω. From the vanishing of the Ricci tensor Ωic, we deduce that $\Omega sc = 0$, and thus, we obtain $Sc = 3\,e^{-2\Theta}\,(1 + 2\,Tr(Q_0))$. We can compute the trace of Q_0 in the basis $\mathfrak{B}$, and then, we have $Tr(Q_0) = \sum_{\alpha=1}^{4} \frac{1}{2}\mathfrak{v}^{*,\alpha}\left(\mathfrak{v}_\alpha - \frac{3}{2}\,\pi(\mathfrak{v}_\alpha)\,\xi\right) = \sum_{\alpha=1}^{4} \frac{1}{2}\left(1 - \frac{3}{2}\,\pi(\mathfrak{v}_\alpha)\,\mathfrak{v}^{*,\alpha}(\xi)\right) = 2 - \frac{3}{4}$, and finally, we obtain

$$Tr(Q_0) = \frac{5}{4}\,. \tag{6.17}$$

Hence, the value of the scalar Riemannian curvature is the following:

$$Sc = \frac{21}{2}\,e^{-2\Theta}\,. \tag{6.18}$$

Remarkably, we obtain a result similar to the one given in the fundamental example (2.12) in introduction. In addition, we deduce also the Ricci tensor Ric associated to g (see Appendix J for the notations and [Kul70, p.318]):

$$Ric(\mathfrak{u},\mathfrak{v}) = \Omega ic(\mathfrak{u},\mathfrak{v}) + 2\,Q(\mathfrak{u},\mathfrak{v}) + g(\mathfrak{u},\mathfrak{v})\left\{3\,\|\varPsi\|^2 + Tr(Q_0)\right\} \tag{6.19}$$

Thus, $Ric(\mathfrak{u},\mathfrak{v}) = 2\,(Q(\mathfrak{u},\mathfrak{v}) + g(\mathfrak{u},\mathfrak{v}))$, and finally we obtain

$$Ric(\mathfrak{u},\mathfrak{v}) = 3\left[g(\underline{\mathfrak{u}},\underline{\mathfrak{v}}) + \frac{1}{2}\,\pi(\mathfrak{u})\,\pi(\mathfrak{v})\right]. \tag{6.20}$$

Furthermore, the Weyl tensor $\mathcal{W}$ associated to g is equal to the Weyl tensor associated with Ω which is equal to Ω itself because $\Omega ic = 0$. Therefore, we have

$$\mathcal{W} = \Omega\,, \qquad\qquad i_\xi\mathcal{W} = 0\,. \tag{6.21}$$

The relation $i_\xi \mathcal{W} = 0$ indicates that $\mathcal{W}$ varies conformally along the time-like curves in $\mathcal{M}_{RPS}$ with tangent vector field ξ. It follows also that the scalar Weyl curvature is vanishing as in the fundamental example (2.12) given in introduction again, and thus, we have the following.

Theorem 9. *The* experienced spacetime $\mathcal{M}_{RPS}$ *is conformally flat.*

In addition, from the definition of S_Θ, we deduce that[37]

$$Tr(S_\Theta) = 4\,\theta\,, \tag{6.22}$$

and thus, $Tr(\Gamma_{\mathfrak{B}''}) = Tr(\omega_{\mathfrak{B}''}) + 4\theta = Tr(\omega_{\mathfrak{B}''}) + 2d\kappa^1$, *i.e.*, the relation

$$Tr(\Gamma_{\mathfrak{B}''}) = d\ln(N(\tau_{(\kappa)})) \tag{6.23}$$

holds. Lastly, from $\Psi \equiv \xi/2$, we have $S_\Theta(\xi, \mathfrak{u}) = (i_\mathfrak{u}\theta)\,\xi + (i_\xi\theta)\,\mathfrak{u} - g(\mathfrak{u}, \xi)\,\Psi = g(\Psi, \mathfrak{u})\,\xi + \frac{1}{2}\,\mathfrak{u} - g(\mathfrak{u}, \xi)\,\Psi = \frac{1}{2}\,\mathfrak{u}$, and thus, $\widetilde{\nabla}_\xi \mathfrak{u} = \nabla_\xi \mathfrak{u} + \frac{1}{2}\,\mathfrak{u}$. Thenceforth, setting $\mathfrak{u} \equiv \mathfrak{v}''_\alpha$, and then, considering that $i_\xi \omega_{\mathfrak{B}''} = -\frac{1}{2}\,\mathbb{1}$, we deduce that $\widetilde{\nabla}_\xi \mathfrak{v}''_\alpha = \nabla_\xi \mathfrak{v}''_\alpha + \frac{1}{2}\,\mathfrak{v}''_\alpha = 0$. Therefore, we obtain:

- $$i_\xi \Gamma_{\mathfrak{B}''} = 0\,, \tag{6.24a}$$
- $$i_\xi \Gamma_{\mathfrak{B}} = \frac{1}{2}\,\mathbb{1}\,. \tag{6.24b}$$

We can conclude with the following remarks: 1) the theorem just above results mainly from the normal projective Cartan connection only, 2) all of the Weyl tensors R, Ric, etc. are algebraic expressions of g and π from only the ***crucial*** condition 2 in Theorem 4, *i.e.*, $\nabla_\mathfrak{u}\xi = \mathfrak{q}_0(\mathfrak{u})$, 3) the Weyl tensor $\mathcal{W}$ is horizontal, *i.e.*, $i_\xi \mathcal{W} = 0$, and 4) we have $\mathcal{L}\,g = \pi \otimes g$ where $\mathcal{L}$ is the Lie derivative on $\mathcal{M}_{RPS}$. Moreover, once g and ξ are given we can deduce a Euclidean metric e such that $e \equiv 2\,\pi \otimes \pi - g$, and, consequently, all of the previous results apply also with e.

All of these conditions can be taken whatever is the dimension of a manifold M satisfying the four conditions of foliation given in Paragraph III C 1 a whatever is its dimension. Hence, we can state the following.

---

[37] Proof: Let $\{Z_1, \ldots, Z_n\}$ be a basis of vector fields and $\{Z^{*1}, \ldots, Z^{*n}\}$ its dual cobasis, then, $Tr(S_\Theta)(\mathfrak{u}) = \sum_{i=1}^n Z^{*i}(S_\Theta(\mathfrak{u}, Z_i)) = \sum_{i=1}^n Z^{*i}((i_\mathfrak{u}\theta)\,Z_i + (i_{Z_i}\theta)\,\mathfrak{u} - g(\mathfrak{u}, Z_i)\,\Psi) = n\,i_\mathfrak{u}\theta + \sum_{i=1}^n \left\{g(Z_i, \Psi)\,Z^{*i}(\mathfrak{u}) - g(\mathfrak{u}, Z_i)\,Z^{*i}(\Psi)\right\} = n\,i_\mathfrak{u}\theta + g(\mathfrak{u}, \Psi) - g(\mathfrak{u}, \Psi) = n\,i_\mathfrak{u}\theta = n\,\theta(\mathfrak{u})$.

Theorem 10. *Let M be a $(n+1)$-dimensional manifold satisfying the four conditions of codimension one foliation given in Paragraph III C 1 a, and such that its n-dimensional leaves $\mathcal{J}$ are modeled on $\mathbb{R}P^n$. Then, there exist a Lorentzian metric g and a Euclidean metric e on M such that*

1. *all of the (co-)tensors U deduced from covariant derivations of the metrics g or e are algebraic (co-)tensors with respect to g or e and the projective form π,*
2. *the Weyl tensor $\mathcal{W}$ defined from g or e is horizontal, i.e., $i_\xi \mathcal{W} = 0$, and $\mathcal{L}_\xi \mathcal{W} = 0$ where $\mathcal{L}$ is the Lie derivative on M, and*
3. *M is conformally flat with respect to both g and e.*

VII. INTERPRETATIONS – GENERAL REMARKS

However, although relativistic positioning systems give right positionings of the spacetime events, the geometrical and physical meaning of the results presented in the precedent sections is that the relativistic positioning systems can only unveil very specific Riemannian structures on the spacetime manifold $\mathcal{M}$ and that they cannot provide complete descriptions of the true underlying Riemannian spacetime manifold $\mathcal{M}$. Indeed, on $\mathcal{M}_{RPS}$, only the projective structure is intrinsic due the arbitrariness in the choice of the time stamps τ_α up to conformal scalings due to the conformal structure. It involves that we need for a physical supplementary process to determine the “true” and “complete” connection $\widehat{\Gamma}$ (or Riemann tensor $\widehat{R}$) of $\mathcal{M}$ and, somehow, escape to the scale arbitrariness of the time stamps. It means the latter are not sufficient to access to more general “nonconstrained” Riemannian structures on $\mathcal{M}$, and therefore, additional physical parameters must be introduced to have a “true” and “complete” description.

Hence, we are faced with a dimensional ambiguity: does the spacetime dimension equal to four or more? Actually, we can say that $\mathcal{M}$ is a manifold but which can be physically investigated as a four dimensional manifold only if embedded in a higher dimensional manifold with additional physical/geometrical parameters (variables) differing possibly from the time stamps τ_α. Besides, giving a conformally equivalent metric $\tilde{g}$ such that $\tilde{g} \equiv e^{2\varphi}\, g$ where $\varphi \in$

$C^2(\mathcal{M}_{RPS})$ does not remove the constraints on the Riemann structures associated with the projective structure on $\mathcal{M}_{RPS}$. And therefore, we obtain only a redefinition of the Yano-Ishihara projecting 1-form π, and then, a conformal factor will not be at the origin of any additional parameter breaking the projective structure.

Apart from these geometrical aspects, we recall what W. Kundt and B. Hoffmann said in their seminal paper on the geometry of spacetime [KH62]:

> *"The fact that the metrical tensor has a gravitational-inertial as well as a metrical significance means that standard length and standard time are [also] determined by the inertial motions of free particles of both non-zero and zero rest masses. To be more precise: the projective structure of space-time given by the (timelike) non-null geodesics, together with the conformal structure given by the null lines determine the metrical structure to within a scale factor that is* independent of position.*"*

In other words, for these authors, that the scale factor "*is independent of position*" may mean that the Riemannian manifold and the Riemannian structure may not only dependent on the underlying spacetime geometry of events as soon as we consider non-geodesic wordlines.

In their seminal paper, these authors shown also explicitly that the dynamics of the physical content must be taking into account and how it burst into the spacetime geometry description. They provide a complex protocol to reach this scaling factor but with no relativistic positioning system "at hand." In the same way, this "dynamics emergence" is explicit in the CFMT protocol with scaling factors depending on the accelerations — and thus applied forces — and the worldlines of both the observers and the emitters (satellites) of the relativistic positioning constellation.

More generally, that the *scale factor is independent of position*, or depends on forces as in the CFMT protocol, this means that we must include certain vectors at each spacetime event τ to obtain a complete Riemannian description of $\mathcal{M}$. Moreover, we can notice the following.

We can never have a conformally flat Riemann manifold $\mathcal{M}$ contrary to $\mathcal{M}_{RPS}$ and lower dimensional cases (cf. CFMT protocol; dimensions ≤ 3) if we consider that $\mathcal{M}$ has no projective structures with respect to $\mathbb{R}P^3$. Indeed, as indicated in introduction, conformal flatness both with a torsion-free Riemann structure can involve the existence of an orthogonal coframe, *i.e.*, a coframe of 1-forms σ_α such that $\sigma_\alpha \wedge d\sigma_\alpha = 0$ for all α and with the metric diagonal with respect to these σ_α's. But, diagonalization is not projectively "stable," *i.e.*, Euclidean reducibility of tensors does not involves a sort of "projective reducibility." Indeed, punctually (*i.e.*, at a given event τ fixed), the orthogonal group or the linear group $GL(4, \mathbb{R})$ does not respect the projective structure contrary to one of its sub-group, namely, the affine group $\mathbb{R}^* \times \mathit{Aff}(3, \mathbb{R})$ keeping invariant the space and time splitting in the tangent or cotangent spaces of $\mathcal{M}_{RPS}$. But, with respect to this affine group we cannot always diagonalize the metric on $\mathcal{M}$ which must differ, somehow, from the metric given on $\mathcal{M}_{RPS}$. Hence, because diagonalization remains impossible if there are given $\mathbb{R}P^3$ projective structures on $\mathcal{M}$, then, conformal flatness on $\mathcal{M}$ is not possible in full generality in this specific case. But, nevertheless, $\mathcal{M}_{RPS}$ can be conformally flat and $\mathcal{M}$ could have a $\mathbb{R}P^4$ projective structure. Roughly speaking, this result originates from the choice of coordinates: homogeneous coordinates on $\mathcal{M}$ and inhomogeneous coordinates on $\mathcal{M}_{RPS}$. Therefore, the actions of the various groups differ strongly.

More precisely, it is usual to reduce a given metric $\eta \equiv \sum_{\alpha,\beta=0}^{3} \eta_{\alpha\beta}\, dx^\alpha \odot dx^\beta$ on $\mathcal{M}$ in such a way to exhibit a horizontal and vertical splitting. We consider that the coordinates x^α are homogeneous coordinates, contrary to those on $\mathcal{M}_{RPS}$. It has been done in various situations such as in the Kundt-Hoffmann protocol or by writing the metric in a synchronous comoving coordinate system exhibiting a Bolyai-Lobachevski projective geometry with a space and time splitting. For instance, let x^i for $i = 1, 2, 3$ the homogeneous horizontal (spatial) coordinates and x^0 the homogeneous (time) coordinate, then, η can put in the following form:

$$\eta = e^{2\vartheta} \left(\sum_{i,j=1}^{3} \mu_{ij}\, dx^i \odot dx^j - d\tilde{x}^0 \odot d\tilde{x}^0 \right), \tag{7.1}$$

where $d\tilde{x}^0 \equiv dx^0 - \sum_{k=1}^{3} \rho_k\, dx^k$ and

$$\eta_{00} \equiv -e^{2\vartheta}, \qquad \rho_k \equiv \frac{\eta_{k0}}{\eta_{00}}, \qquad \rho_{ij} \equiv \frac{\eta_{ij}}{\eta_{00}}, \qquad \mu_{ij} \equiv \rho_i\, \rho_j - \rho_{ij}\,. \tag{7.2}$$

Thenceforth, let $\{\partial_{\tilde{x}^0}, \partial_{x^1}, \partial_{x^2}, \partial_{x^3}\}$ be the dual frame of the coframe $\{d\tilde{x}^0, dx^1, dx^2, dx^3\}$, then, a change of homogeneous coordinates leaving invariant the horizontal representing manifold defined by $\tilde{x}^0 = cst$ (*i.e.*, $\tilde{x}^0$ is therefore the vertical homogeneous coordinate) is such that

$$\begin{aligned} &\partial_{\tilde{y}^0} \equiv \partial_{\tilde{x}^0}\,, \\ &\partial_{y^i} \equiv \sum_{j=1}^{3} a_i^j\,\partial_{x^j} + a_i^0\,\partial_{\tilde{x}^0}\,. \end{aligned} \tag{7.3}$$

Then, we obtain

$$\begin{aligned} &d\tilde{x}^0 \equiv d\tilde{y}^0 + \sum_{j=1}^{3} a_j^0\,dy^j\,, \\ &dx^i \equiv \sum_{j=1}^{3} a_i^j\,dy^j\,. \end{aligned} \tag{7.4}$$

This transformation represents the action of an element of the affine group $\mathbb{R}^*\times \mathit{Aff}(3,\mathbb{R})$ acting on $\mathcal{M}$. It is then obvious that the Euclidean reducibility is not kept under such change of homogeneous coordinates meanwhile the horizontal and vertical splitting remains invariant. Indeed, on the contrary, we should obtain a representation of η such that $\eta \equiv e^{2\vartheta}\left(\sum_{i,j=1}^{3} \nu_{ij}\,dy^i \odot dy^j\right.$ $\left.-d\tilde{y}^0 \odot d\tilde{y}^0\right)$. The result would be completely different if the coordinates x^α were inhomogenous coordinates, *i.e.*, if we were on $\mathcal{M}_{RPS}$ since, in this case, the conformal action of $\mathbb{R}^*\times \mathit{Aff}(3,\mathbb{R})$ involves $a_i^0 \equiv 0$.

Hence, the Euclidean reducibility is not a suitable concept in projective geometry. That is the main and important difference between Euclidean tensors and projective tensors which are of a structural affine nature. It has been historically pointed out by É. Cartan in a unique (to the author's knowledge) paper [Car35, in french] on that subject and along an approach differing strongly from those of O. Veblen, B. Hoffmann, T.Y. Thomas, J.M. Thomas, D.J. Struik, and J.A. Schouten who never noticed this fundamental and basic discrepancy. Somehow, these authors shown indirectly their troubles in the symbolic mathematical notations they used. For instance, J.A. Schouten used the notation "$\overset{*}{=}$" with the following comment: "*Le signe* $\overset{*}{=}$ *signifie que l'égalité n'est valable que pour le ou les systèmes de coordonnées pour lesquelles elle est effectivement écrite.*" ("The sign $\overset{*}{=}$ means that the equality is only valid for coordinate system(s) for which it is actually written.").

This explains why we can never diagonalize the metric on $\mathcal{M}$ or reduced it into two separate horizontal and vertical parts stable with respect to $\mathbb{R}^*\times \mathit{Aff}(3,\mathbb{R})$.

Nevertheless, since a conformally flat spacetime manifold $\mathcal{M}$ can exist then only a spacetime manifold $\mathcal{M}$ endowed with a $\mathbb{R}P^4$ projective structure can be admitted; and thus a spacetime manifold $\mathcal{M}$ modeled on $\mathbb{R}P^4$.

Now, precisely, we present results on a complementary physical protocol in $\mathcal{M}$ using RPSs and from which $\mathcal{M}$ inherits necessarily a $\mathbb{R}P^4$ projective structure to be completely observable and finally geometrically reached.

VIII. A PROTOCOL IMPLEMENTED BY RECEIVERS TO LOCALIZE EVENTS

Following the terminology of B. Coll, a RPS is 'primary' if it satisfies the three following criteria: it is 1) 'generic,' *i.e.*, the system of coordinates it provides must exist independently of the spacetime geometry for each given class of spacetime, 2) it is 'free,' *i.e.*, its structure does not need the knowledge of the gravitational field, and 3) it is 'immediate,' *i.e.*, the receivers know their positions without delays at the instant they receive the four time stamps τ_α sent by the four emitting satellites of the RPS constellation. The RPS we present designed by CFMT, namely, the SYPOR ("*SYstème de POsitionnement Relativiste*"), belongs to this category, but we ask for a supplementary protocol to allow any receiver to locate any event in his surrounding and in the spacetime region covered by the RPS. The goal for seeking such a tracking protocol is to find a way to break the scaling indeterminacy leading to a projective description of the spacetime geometry, and in return, to have access to the "true" Riemannian four dimensional spacetime structure on $\mathcal{M}$. This kind of protocol can be called a *relativistic stereometric protocol* [Col13].

In this section, we present such a protocol. It has two major flaws which we nevertheless think that they are unavoidable: its implementation is complex and it may be immediate only in some very particular situations or regions covered by the RPS depending of the located event. In full generality, obviously, it cannot be immediate because the satellites of any constellation must "wait" the signals coming from the source event which will be afterward localized. But, nevertheless, it really breaks the scaling indeterminacy and provides an access to $\mathcal{M}$ as expected. Moreover, it possibly gives a completely new interpretation of a particular sort of so-called

Weyl's length connection which could circumvent, by construction, the fundamental Einstein criticisms.

A. The protocol of localization in a $(1+1)$-dimensional spacetime $\mathcal{M}$

In this situation , the protocol is very simple. We recall, first, the principle for the relativistic positioning. For, we consider two emitters, namely, $\mathcal{E}_1$ and $\mathcal{E}_2$ and a user $\mathcal{U}$ with their respective (time-like) worldlines W_1, W_2 and $W_{\mathcal{U}}$. The two emitters broadcast their two time stamps τ_1 and τ_2 generated by embarked clocks, and then, the two-dimensional grid can be constructed from this RPS. From a system of echoes (Figure VIII.1), the user at the events $U_1 \in W_{\mathcal{U}}$ and $U_2 \in W_{\mathcal{U}}$ receives four numbers: (τ_1^+, τ_2^-) from E_1 and (τ_1^-, τ_2^+) from E_2 (see Figure VIII.2). In addition, from this RPS, the user can also know in this grid the two events E_1 and E_2 at which the two emitters sent these four time stamps *viz*, $E_1 \equiv (\tau_1^+, \tau_2^-)$ and $E_2 \equiv (\tau_1^-, \tau_2^+)$.

Then, let e be an event in the hexagonal domain (see Figure I.2). This event can be at the intersection point of the two light rays received by $\mathcal{E}_1$ and $\mathcal{E}_2$ at the events E_1 and E_2 (see Figure VIII.2). Hence, the position of e in the grid is easily deduced by $\mathcal{U}$ if 1) $\mathcal{U}$ records (τ_1^+, τ_2^-) and (τ_1^-, τ_2^+) along $W_{\mathcal{U}}$, and 2) a physical identifier for e is added at E_1 and E_2 to each pair of time stamps to be matched by $\mathcal{U}$. Nevertheless, the localization of e cannot be reached if e is outside the hexagonal domain.

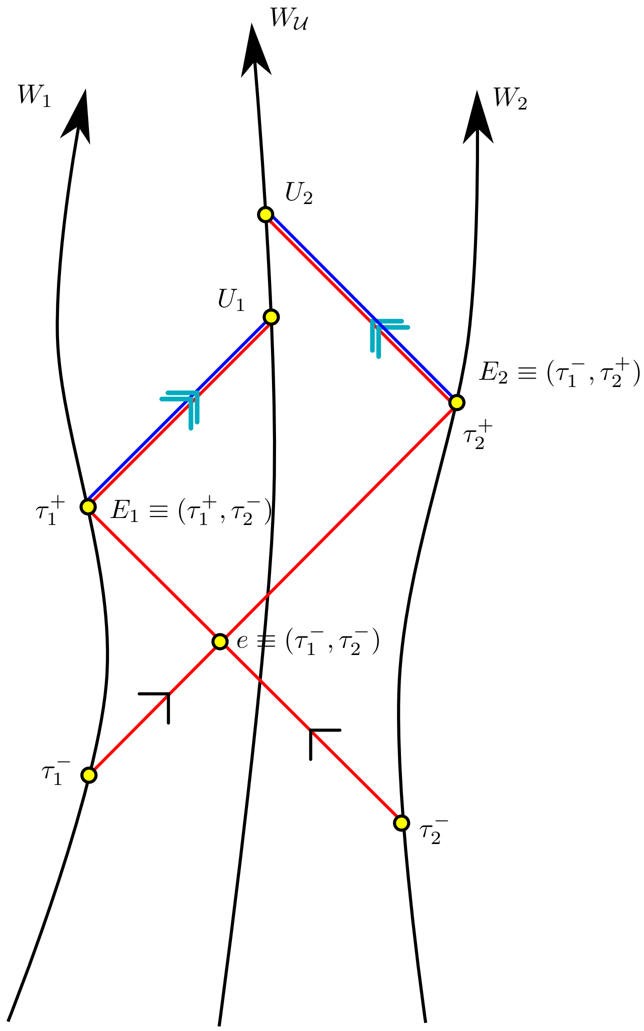


Figure VIII.1. The system of echoes in dimension two.

B. The protocol of localization in a $(2+1)$-dimensional spacetime $\mathcal{M}$ modeled on $\mathbb{R}P^3$

In this case, the complexity of the protocol increases dramatically. Again, we consider three emitters $\mathcal{E}$, $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ broadcasting three sets of time stamps denoted, respectively, by τ, $\tilde{\tau}$ and $\hat{\tau}$. Then, the grid is the Euclidean space $\mathbb{R}^3$ with the system of Cartesian coordinates $(\tau, \tilde{\tau}, \hat{\tau})$. Then, we consider, first, the system of echoes from $\mathcal{E}$ to the user $\mathcal{U}$. This system can be outlined as indicated on the figure VIII.3. Then, the user receives at the reception event U seven time stamps sent by $\mathcal{E}$ and emitted at the event of emission $E \in W$, where W is the worldline of

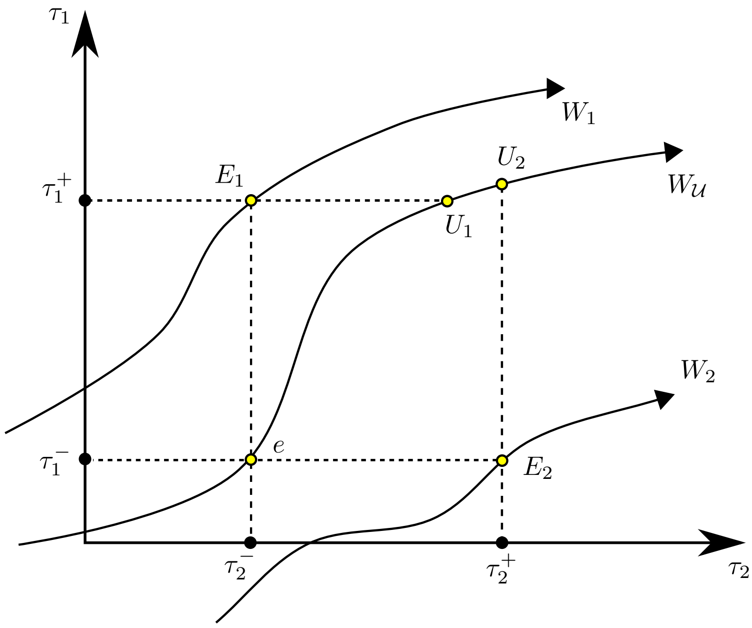


Figure VIII.2. The two-dimensional grid.

$\mathcal{E}$: $(\tau_1, (\tilde{\tau}_1^\tau, \tilde{\tau}_2^\tau, \tilde{\tau}_3^\tau), (\hat{\tau}_1^\tau, \hat{\tau}_2^\tau, \hat{\tau}_3^\tau))$. Besides, the emitter $\mathcal{E}$ receives at E six time stamps from the other two emitters $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$, *viz*, $p_{\widetilde{E}'} \equiv (\tilde{\tau}_1^\tau, \tilde{\tau}_2^\tau, \tilde{\tau}_3^\tau)$ emitted at $\widetilde{E}'$ from $\widetilde{\mathcal{E}}$, and $p_{\widehat{E}'} \equiv (\hat{\tau}_1^\tau, \hat{\tau}_2^\tau, \hat{\tau}_3^\tau)$ emitted at $\widehat{E}'$ from $\widehat{\mathcal{E}}$. Actually, $p_{\widetilde{E}'}$ and $p_{\widehat{E}'}$ are the 3-positions of, respectively, $\widetilde{E}'$ and $\widehat{E}'$ in the grid. Moreover, $\mathcal{E}$ sends at E the time stamp τ_1 received at U by the user.

In addition, two of the three time stamps received at $\widetilde{E}'$ are sent by $\mathcal{E}$ at E': $\tilde{\tau}_1^\tau$, and by $\widehat{\mathcal{E}}$ at $\widehat{E}''$: $\tilde{\tau}_3^\tau$; and we have a similar situation for $\widehat{E}'$ (see Figure VIII.3).

Then, the user can deduce the 3-position p_E of the event E in the grid: $p_E \equiv (\tau_1, \tau_2, \tau_3) \equiv (\tau_1, \tilde{\tau}_2^\tau, \hat{\tau}_3^\tau)$, and the two 3-positions $p_{\widetilde{E}'}$ and $p_{\widehat{E}'}$ of the two events $\widetilde{E}'$ and $\widehat{E}'$ respectively. Additionally, $\tilde{\tau}_2^\tau$ is emitted by $\widetilde{\mathcal{E}}$ at $\widetilde{E}'$, and $\hat{\tau}_3^\tau$ is emitted by $\widehat{\mathcal{E}}$ at $\widehat{E}'$. Also, these two 3-positions are obtained from four time stamps emitted from four events, namely, E' and $\widehat{E}''$ for $\widetilde{E}'$, and E'' and $\widetilde{E}''$ for $\widehat{E}'$ (see Figure VIII.3).

Actually, the user receives 3×7 time stamps, *i.e.*, three sets of data, namely, d_E, $d_{\widetilde{E}}$ and

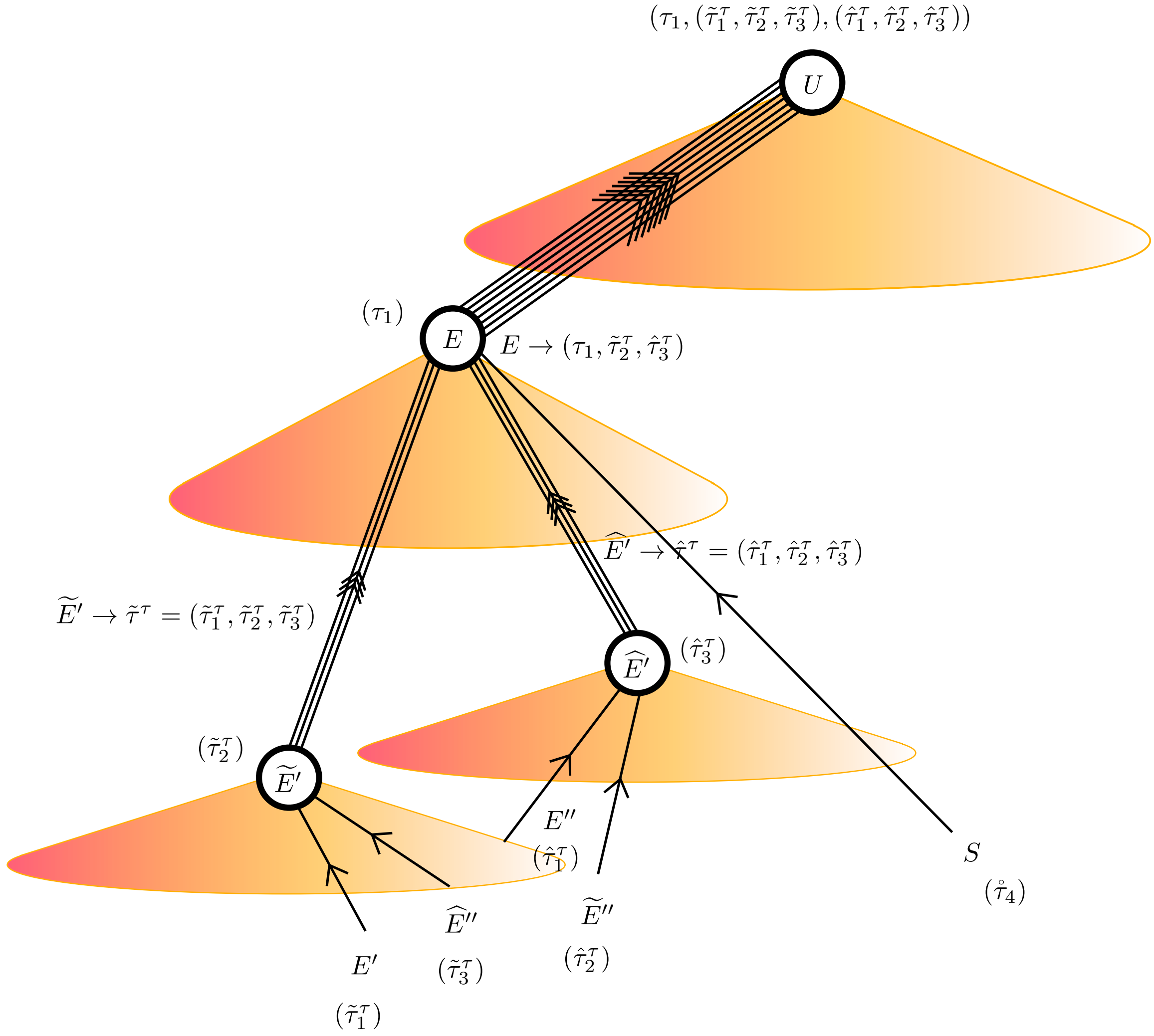


Figure VIII.3. The system of echoes with four past null cones.

$d_{\widehat{E}}$ such that

$$d_E \equiv (\tau_1, (\tilde{\tau}_1^\tau, \tilde{\tau}_2^\tau, \tilde{\tau}_3^\tau), (\hat{\tau}_1^\tau, \hat{\tau}_2^\tau, \hat{\tau}_3^\tau), id_{\mathcal{E}}) \quad \text{received at} \quad U \in {}^{\mathcal{U}}W\,,$$
$$d_{\widetilde{E}} \equiv (\tilde{\tau}_2, (\hat{\tau}_1^{\tilde{\tau}}, \hat{\tau}_2^{\tilde{\tau}}, \hat{\tau}_3^{\tilde{\tau}}), (\tau_1^{\tilde{\tau}}, \tau_2^{\tilde{\tau}}, \tau_3^{\tilde{\tau}}), id_{\widetilde{\mathcal{E}}}) \quad \text{received at} \quad \widetilde{U} \in {}^{\mathcal{U}}W\,,$$
$$d_{\widehat{E}} \equiv (\hat{\tau}_3, (\tau_1^{\hat{\tau}}, \tau_2^{\hat{\tau}}, \tau_3^{\hat{\tau}}), (\tilde{\tau}_1^{\hat{\tau}}, \tilde{\tau}_2^{\hat{\tau}}, \tilde{\tau}_3^{\hat{\tau}}), id_{\widehat{\mathcal{E}}}) \quad \text{received at} \quad \widehat{U} \in {}^{\mathcal{U}}W\,,$$

where ${}^{\mathcal{U}}W$ is the worldline of the user and $id_{\mathcal{E}}$, $id_{\widetilde{\mathcal{E}}}$ and $id_{\widehat{\mathcal{E}}}$ are identifiers of the emitters (see Figure VIII.4). From now, we consider only the sets of events represented on the figure VIII.3.

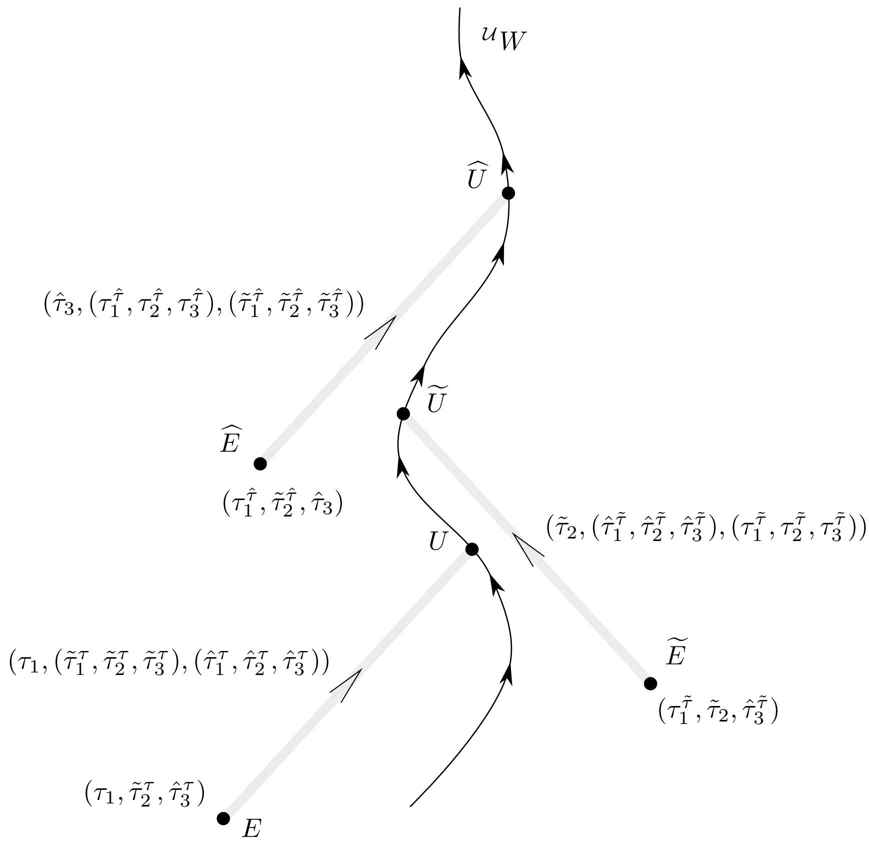


Figure VIII.4. The three data received and recorded by the user at U, $\widetilde{U}$ and $\widehat{U}$.

Then, the user can also deduce three future light-like vectors generating the future null cone at E, namely, $\hat{k}_E$, $\tilde{k}_E$ and k_E^U such that

$$\hat{k}_E \equiv \overrightarrow{E\widehat{P}_E} \equiv p_E - p_{\widehat{E}'}\,, \quad \tilde{k}_E \equiv \overrightarrow{E\widetilde{P}_E} \equiv p_E - p_{\widetilde{E}'}\,, \quad k_E^U \equiv \overrightarrow{EP_E^U} \equiv p_U - p_E\,,$$

where $P_E^U \equiv U$ and p_U is the 3-position of U in the grid. The three ending points $\widehat{P}_E$, $\widetilde{P}_E$ and P_E^U define an affine plane A_E in the grid. Then, a unique circumcircle in A_E contains these three ending points from which the unique circumcenter $C \in A_E$ can be deduced by standard formulas.[38]

Then, let e be an event to localize in the grid. It is featured and identified by a set s_e

[38] For, we define the two relative vectors with origin U: $\tilde{r} = \tilde{k}_E - k_E^U$ and $\hat{r} = \hat{k}_E - k_E^U$. Then, in $\mathbb{R}^3$, the circumcenter C is the point $C \in A_E$ such that

$$\overrightarrow{UC} = \frac{(\|\tilde{r}\|^2\,\hat{r} - \|\hat{r}\|^2\,\tilde{r}) \wedge (\tilde{r} \wedge \hat{r})}{2\,\|\tilde{r} \wedge \hat{r}\|^2}\,.$$

of physical, non-geometrical characteristics such as, for instance, its spectrum, its shape, its temperature, etc. We assume also that this event e can be detected and almost instantaneously physically analyzed by the emitters at the events E, $\widetilde{E}$ and $\widehat{E}$ from signals carried by light rays, for instance, coming from e. Also, we consider that these light rays, carrying these various physical informations, manifest in "bright points" on their respective "celestial circles." For the sake of illustration, we consider only the celestial circle $\mathcal{C} \simeq S^1$ of the emitter $\mathcal{E}$ at the event E. Also, we provide $\mathcal{E}$ with an apparatus made of an optical device and a compass to locate the event e on the celestial circle $\mathcal{C}$.[39] For, we need also to define a projective frame for $\mathcal{C}$. For this purpose, the two other satellites $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ manifest in "bright points" on $\mathcal{C}$ ascribed to the two events $\widetilde{E}'$ and $\widehat{E}'$ in the past null cone of E. Then, the projective point $[0]_E \in \mathcal{C}$ is ascribed to $\widetilde{E}'$ and $\tilde{k}_E$, and the projective point $[\infty]_E \in \mathcal{C}$ is ascribed to $\widehat{E}'$ and $\hat{k}_E$:

$$\widetilde{E}' \longleftrightarrow [0]_E \longleftrightarrow \tilde{k}_E \,,$$
$$\widehat{E}' \longleftrightarrow [\infty]_E \longleftrightarrow \hat{k}_E \,.$$

Then, we consider that $\mathbb{R}P^1 \simeq \mathcal{C} = S^1$. Note that we cannot ascribe to k_E^U and U a projective point $[1]_E \in \mathcal{C}$ since U is in the future null cone of E. Therefore, we need a fourth satellite, namely, $\mathcal{S}$ in addition to $\mathcal{E}$, $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$. A priori, $\mathcal{S}$ does not need to broadcast a supplementary time stamp, but it must be clearly identified with an identifier $id_{\mathcal{S}}$. Then, another fourth "bright point" ascribed to the third projective point $[1]_E \in \mathcal{C}$ is observable on $\mathcal{C}$ due to $\mathcal{S}$ sending its identifier $id_{\mathcal{S}}$ from the event S:

$$S \longleftrightarrow [1]_E \,.$$

Now, e can be localized applying the following first procedure.

From the "bright points" $[\infty]_E$, $[0]_E$ and $[1]_E$ and the optical device and compass embarked on $\mathcal{E}$, the optical observation of e on $\mathcal{C}$ provides a projective point $[\alpha]_E \in \mathcal{C}$ with α clearly,

[39] The only remaining step utilizing material objects is the angle measurement by compasses. Their use implies that the angles remain invariant regardless of the size of the compass. And then, this also implies that there is an absolute notion of angle in contrast to the notions of time and length which depend on frames. This has historically been regarded by H. Weyl and K. Gödel with their concepts of 'inertial compass' or 'star compass' in objection with Mach's principle. This absolute feature cannot come from any geometry of space-time. It is therefore possible that it comes from a different physics like quantum mechanics. Thus, a true compass would be based on the use of a quantum phenomenon of angle measurement. This can be done with a Michelson interferometer (see for example [Sch54, She09]). Nevertheless, we think that the compass should be rather graduated by fractional numbers such as those appearing in the fractional Hall effect for instance.

numerically evaluated from the projective frame $\mathfrak{F}_E \equiv \{[\infty]_E, [0]_E, [1]_E\}$. Moreover, to $[\alpha]_E$ corresponds two vectors $\vec{v}_E^{\,+}$ and $\vec{v}_E^{\,-}$ such that

$$\vec{v}_E^{\,\pm} \equiv \overrightarrow{EV_E^{\pm}} \equiv \overrightarrow{EC} \pm \left(\overrightarrow{C\widetilde{P}_E} + \alpha\, \overrightarrow{C\widehat{P}_E}\right),$$

where, in addition, $\overrightarrow{C\widetilde{P}_E}$ and $\overrightarrow{C\widehat{P}_E}$ are ascribed to the following projective points:

$$\begin{aligned}
&\overrightarrow{C\widetilde{P}_E} \longleftrightarrow [0]_E\,,\\
&\overrightarrow{C\widehat{P}_E} \longleftrightarrow [\infty]_E\,.
\end{aligned}$$

Now, the two vectors $\vec{v}_E^{\,\pm}$ define a two dimensional affine plane $\mathcal{P}_e$ containing e such that

$$\overrightarrow{Ee} = a^+\, \vec{v}_E^{\,+} + a^-\, \vec{v}_E^{\,-} \in \mathcal{P}_e$$

for two reals $a^\pm$ to be determined applying the same procedure with the two emitters $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ at, respectively, $\widetilde{E}$ and $\widehat{E}$. Indeed, we deduce the two other analogous affine plane $\widetilde{\mathcal{P}}_e$ and $\widehat{\mathcal{P}}_e$ and two relations such as

$$\begin{aligned}
&\overrightarrow{\widetilde{E}e} = \tilde{a}^+\, \vec{\tilde{v}}_E^{\,+} + \tilde{a}^-\, \vec{\tilde{v}}_E^{\,-} \in \widetilde{\mathcal{P}}_e\,,\\
&\overrightarrow{\widehat{E}e} = \hat{a}^+\, \vec{\hat{v}}_E^{\,+} + \hat{a}^-\, \vec{\hat{v}}_E^{\,-} \in \widehat{\mathcal{P}}_e\,.
\end{aligned}$$

And then, e is the intersection point of $\mathcal{P}$, $\widetilde{\mathcal{P}}_e$ and $\widehat{\mathcal{P}}_e$. Therefore, we obtain six algebraic linear equations determining completely the a's and then e in the grid.

But, a second simpler procedure can be applied using again optical devices and compasses. It is based on a change of projective frame in $\mathcal{C}$. More precisely, in the previous procedure with the projective frame $\mathfrak{F}_E$ at E, the three projective points $[\infty]_E$, $[0]_E$ and $[1]_E$ defining $\mathfrak{F}_E$ were ascribed to, respectively, $\widehat{E}'$, $\widetilde{E}'$ and S. Now, we consider another projective frame $\mathfrak{F}'_E \equiv \{[\infty]'_E, [0]'_E, [1]'_E\}$ such that the following correspondence

$$\begin{aligned}
\widetilde{E}' &\longleftrightarrow [\tilde{\tau}_1^\tau]'_E\,,\\
\widehat{E}' &\longleftrightarrow [\hat{\tau}_1^\tau]'_E\,,\\
S &\longleftrightarrow [\mathring{\tau}_4]'_E
\end{aligned}$$

holds, assuming now that $\mathcal{S}$ broadcast also a fourth time stamp $\mathring{\tau}_4$ in addition to the three time stamps τ_1, τ_2 and τ_3 (see Figure VIII.3). Moreover, in a similar way, each other emitter $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ receives, respectively, at $\widetilde{E}$, the time stamp $\tilde{\mathring{\tau}}_4$ and, at $\widehat{E}$, the time stamp $\hat{\mathring{\tau}}_4$, from $\mathcal{S}$ at two events in ${}^{\mathcal{S}}W$ differing in full generality from the event $S \in {}^{\mathcal{S}}W$. Hence, there are three corresponding emission events on the worldline of $\mathcal{S}$ for these three supplementary time stamps $\mathring{\tau}_4$. Then, to e, it corresponds also another projective point $[\tau_e]'_E$ with respect to this new projective frame $\mathfrak{F}'_E$. Then, the following correspondences

$$
\begin{aligned}
[0]_E &\longleftrightarrow [\tilde{\tau}_1^\tau]'_E\,,\\
[\infty]_E &\longleftrightarrow [\hat{\tau}_1^\tau]'_E\,,\\
[1]_E &\longleftrightarrow [\mathring{\tau}_4]'_E\,,\\
[\alpha_e]_E &\longleftrightarrow [\tau_e]'_E
\end{aligned}
$$

define the change of projective frame and $[\tau_e]'_E$ (see Figure VIII.5).

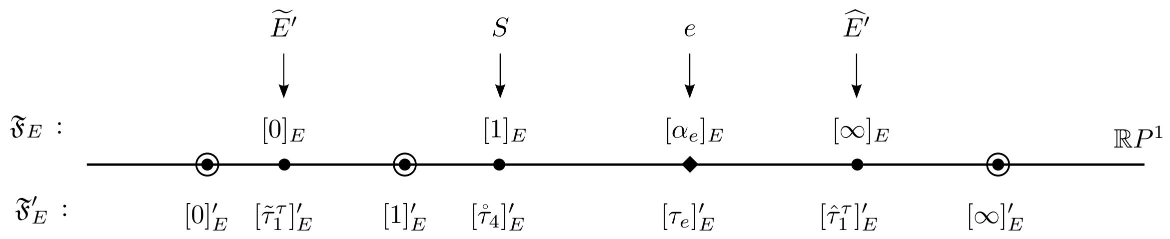


Figure VIII.5. The change of projective frame at E.

In homogeneous coordinates, this change of projective frame is defined by a matrix $K \in GL(2,\mathbb{R})$ such that

$$
K \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix},
$$

and the four following additional correspondences:

$$[0]_E \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{K} \begin{pmatrix} a \\ c \end{pmatrix} \equiv [\tilde{\tau}_1^\tau]'_E \quad \text{where } \tilde{\tau}_1^\tau = a/c\,,$$

$$[\infty]_E \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{K} \begin{pmatrix} b \\ d \end{pmatrix} \equiv [\hat{\tau}_1^\tau]'_E \quad \text{where } \hat{\tau}_1^\tau = b/d\,,$$

$$[1]_E \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix} \xrightarrow{K} \begin{pmatrix} a+b \\ c+d \end{pmatrix} \equiv [\mathring{\tau}_4]'_E \quad \text{where} \quad \mathring{\tau}_4 = \left(\frac{a+b}{c+d}\right),$$

$$[\alpha_e]_E \equiv \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \xrightarrow{K} \begin{pmatrix} \alpha_e\, a+b \\ \alpha_e\, c+d \end{pmatrix} \equiv [\tau_e]'_E \quad \text{where} \quad \tau_e = \left(\frac{\alpha_e\, a+b}{\alpha_e\, c+d}\right).$$

Therefore, we obtain

$$\begin{cases} a = -\,\hat{\tau}_1^\tau\,[\,\tilde{\tau}_1^\tau : \hat{\tau}_1^\tau : \mathring{\tau}_4\,]\,d\,, \\ b = \tilde{\tau}_1^\tau\, d\,, \\ c = [\,\tilde{\tau}_1^\tau : \hat{\tau}_1^\tau : \mathring{\tau}_4\,]\,d\,, \end{cases}$$

where $[\,\tilde{\tau}_1^\tau : \hat{\tau}_1^\tau : \mathring{\tau}_4\,]$ is such that

$$[\,\tilde{\tau}_1^\tau : \hat{\tau}_1^\tau : \mathring{\tau}_4\,] \equiv \left(\frac{\tilde{\tau}_1^\tau - \mathring{\tau}_4}{\hat{\tau}_1^\tau - \mathring{\tau}_4}\right).$$

Then, we deduce τ_e such that

$$\tau_e \equiv \left(\frac{\tilde{\tau}_1^\tau - \alpha_e\,\hat{\tau}_1^\tau\,[\,\tilde{\tau}_1^\tau : \hat{\tau}_1^\tau : \mathring{\tau}_4\,]}{1-\alpha_e\,[\,\tilde{\tau}_1^\tau : \hat{\tau}_1^\tau : \mathring{\tau}_4\,]}\right). \tag{8.1}$$

This is a birational continuous function, and thus bijective. In particular, if $\alpha_e = 0$, 1 or ∞, then we find $\tau_e = \tilde{\tau}_1^\tau$, $\mathring{\tau}_4$ or $\hat{\tau}_1^\tau$. From the other emitters at $\widetilde{E}$ and $\widehat{E}$, the user can compute the three time stamps $p_e \equiv (\tau_e, \tilde{\tau}_e, \hat{\tau}_e)$ ascribed to the 3-position p_e of the event e in the grid; therefore localized as expected. Also, it is important to note that given E, $\widetilde{E}$ and $\widehat{E}$, the event e is *unique* since it is the intersection point of three two-dimensional past null cones. Moreover, we can say that there exists a unique set of three events E, $\widetilde{E}$ and $\widehat{E}$ "attached" to e, *i.e.*, we have a fibered product of past null cones (over the set of localized events in $\mathcal{M}$) homeomorphic to $\mathcal{M}$.

Hence, we need four satellites $\mathcal{E}$, $\widetilde{\mathcal{E}}$, $\widehat{\mathcal{E}}$ and $\mathcal{S}$ with their four time stamps to localize an event in the grid, and thus, the three dimensional spacetime $\mathcal{M}$ must be embedded in $\mathbb{R}^4$. For instance, we have the following coordinates in $\mathbb{R}^4$:

$$E \longleftrightarrow (\tau_1, \tilde{\tau}_2^{\tau}, \hat{\tau}_3^{\tau}, \mathring{\tau}_4)\,, \tag{8.2a}$$

$$\widetilde{E} \longleftrightarrow (\tau_1^{\tilde{\tau}}, \tilde{\tau}_2, \hat{\tau}_3^{\tilde{\tau}}, \tilde{\mathring{\tau}}_4)\,, \tag{8.2b}$$

$$\widehat{E} \longleftrightarrow (\tau_1^{\hat{\tau}}, \tilde{\tau}_2^{\hat{\tau}}, \hat{\tau}_3, \hat{\mathring{\tau}}_4)\,. \tag{8.2c}$$

Also, the data sent by the satellites $\mathcal{E}$, $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ to the user are reduced. We just need the following

$$\bar{d}_E \equiv ((\tau_1, \tilde{\tau}_2^{\tau}, \hat{\tau}_3^{\tau}, \mathring{\tau}_4), id_{\mathcal{E}}, \alpha_e, s_e)\,,$$

$$\bar{d}_{\widetilde{E}} \equiv ((\tau_1^{\tilde{\tau}}, \tilde{\tau}_2, \hat{\tau}_3^{\tilde{\tau}}, \tilde{\mathring{\tau}}_4), id_{\widetilde{\mathcal{E}}}, \tilde{\alpha}_e, s_e)\,,$$

$$\bar{d}_{\widehat{E}} \equiv ((\tau_1^{\hat{\tau}}, \tilde{\tau}_2^{\hat{\tau}}, \hat{\tau}_3, \hat{\mathring{\tau}}_4), id_{\widehat{\mathcal{E}}}, \hat{\alpha}_e, s_e)\,,$$

where s_e allows to match both together the three first data ascribed to e.

Besides, the question rises to know if a fourth coordinate $\mathring{\tau}_{4,e}$ can be ascribed also to the event e as for the three events E, $\widetilde{E}$ and $\widehat{E}$. A coordinate $\mathring{\tau}_{4,e}$ could be easily obtained from the 3-position of e in the grid if 1) e is in the *future horismos* [KP67, GPS05] of a point p on the worldline of $\mathcal{S}$, and 2) $\mathcal{S}$ broadcast also, in particular to the user, the coordinates of p in the grid obtained from the three other emitters $\mathcal{E}$, $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$. The first condition cannot always be physically or technologically satisfied since there necessarily exists an origin event o at which the fourth satellite $\mathcal{S}$ begins to run. Hence, we can expect to know the positions of $\mathcal{S}$ in the grid only beyond this starting point o on the future worldline ${}^{\mathcal{S}}W_o^+ \equiv \{o \ll p, \textit{where } p \textit{ is an emission event of } \mathcal{S}\}$ of $\mathcal{S}$ contained in the chronological future of o. Nevertheless, it is easy to circumvent this difficulty assuming that we define the prolongation ${}^{\mathcal{S}}W_o^-$ of the worldline of $\mathcal{S}$ in the causal past of o by a given, arbitrary, nevertheless well-defined by geometric conventions, curve in the grid. Then, from a given time parameterization of ${}^{\mathcal{S}}W_o^-$, we can also ascribe to any event e a fourth time stamp $\tau_{4,e}$ from the message function $f^-_{{}^{\mathcal{S}}W_o^-} : e \longrightarrow \mathring{\tau}_{4,e}$ (see Introduction I, p.1). Then, the worldline ${}^{\mathcal{S}}W$ of $\mathcal{S}$ is such that ${}^{\mathcal{S}}W = {}^{\mathcal{S}}W_o^- \cup \{o\} \cup {}^{\mathcal{S}}W_o^+$ and we obtain the complete message function $f^-_{{}^{\mathcal{S}}W} : e \in \mathcal{M} \longrightarrow \mathring{\tau}_{4,e} \in \mathbb{R} \simeq {}^{\mathcal{S}}W$. As a consequence, from $f^-_{{}^{\mathcal{S}}W}$, we obtain an embedding of $\mathcal{M}$ in $\mathbb{R}^4$. This embedding is explicit since we cannot localize events without giving a fourth time stamp such as, for instance, $\tilde{\mathring{\tau}}_4$. Nevertheless, as we show in details in the sequel within the context of a $(3+1)$-dimensional spacetime, this is not really the right way to obtain a fourth time stamp although the message functions intervene as well as the worldline $\mathcal{S}_W$.

Furthermore, we recall that we have a local chart $\mu : (\alpha_e, \tilde{\alpha}_e, \hat{\alpha}_e) \in (\mathbb{R}P^1)^3 \longrightarrow p_e = (\tau_e, \tilde{\tau}_e, \hat{\tau}_e) \in \mathbb{R}^3$, and we consider now the action of $PGL(4, \mathbb{R})$ on the triplets $(\alpha_e, \tilde{\alpha}_e, \hat{\alpha}_e)$ of angles. Before, we denote by α_i $(i = 1, 2, 3)$ the three angles such that $\alpha_e \equiv \alpha_1$, $\tilde{\alpha}_e \equiv \alpha_2$ and $\hat{\alpha}_e \equiv \alpha_3$, and by τ_j $(j = 1, 2, 3)$ the three time stamps such that $\tau_e \equiv \tau_1$, $\tilde{\tau}_e \equiv \tau_2$ and $\hat{\tau}_e \equiv \tau_3$. And then, we make below the list of formulas we start with. In particular, we have a first set of formulas from the formulas such as (8.1) at $E \equiv E_1$, $\widetilde{E} \equiv E_2$ and $\widehat{E} \equiv E_3$:

$$\tau_i = \left(\frac{u_i^Q\, \alpha_i + v_i^Q}{w_i^\ell\, \alpha_i + k_i^\ell} \right) \quad \text{at} \quad E_i\,, \tag{8.3}$$

where we assume $w_i^\ell \neq 0$ and where the superscripts Q and ℓ indicate, respectively, that u_i^Q, v_i^Q, w_i^ℓ and k_i^ℓ are homogeneous polynomial of degrees 2 ($Q \equiv$ quadratic) and homogeneous polynomials of degrees 1 ($\ell \equiv$ linear) with respect to the set of time stamps collected at the three E_i for the localization of e. Also, we consider that $P \in PGL(4, \mathbb{R})$ acts on the three angles α_i to give the three angles α'_j such that

$$\alpha_i = \left(\frac{\sum_{j=1}^3 P_i^j\, \alpha'_j + P_i^4}{\sum_{k=1}^3 P_4^k\, \alpha'_k + P_4^4} \right). \tag{8.4}$$

Then, substituting the three angles α_i in the formulas (8.3), we obtain the following second set of formulas:

$$\tau_i = \left(\frac{\sum_{j=1}^3 K_i^j\, \alpha'_j + K_i^4}{\sum_{k=1}^3 H_i^k\, \alpha'_k + H_i^4} \right), \tag{8.5}$$

where the coefficients K_b^a and H_b^a $(a, b = 1, \ldots, 4)$ are linear with respect to the coefficients of $P \equiv (P_b^a)$. But, these formulas can be rewritten in the following general forms:

$$\tau_i = \left(\frac{p_i^Q\, \alpha'_j + q_i^Q}{r_i^\ell\, \alpha'_k + s_i^\ell} \right), \tag{8.6}$$

which are of the same forms as (8.3). In other words, the projective transformation P provides admissible changes of projective frames from $\mathfrak{F}_{E_i}$ to $\mathfrak{F}^{\bullet}_{E_i}$ on the celestial circles at the events E_i. These changes of projective frames are defined from the whole of the time stamps collected at the three events E_i and not only at a particular one. Thus, these changes differ from those from which we obtained the formulas (8.3) for instance. As a consequence, the coefficients p_i^Q, q_i^Q, r_i^ℓ and s_i^ℓ depend on all of the time stamps and not only of those collected at the event E_i. In addition, because we obtain admissible changes of frames, then P is an admissible transformation which can be, therefore, applied on the complete set of angles, *viz*, the set of angles $(\alpha'_1, \alpha'_2, \alpha'_3)$ in the present case or the set of angles $(\alpha_1, \alpha_2, \alpha_3)$ as well.

But, remarkably, the (non-unique) element $P \in PGL(4, \mathbb{R})$ such that, for instance,

$$P_a^a = P_4^i = P_3^4 = 1\,, \qquad a = 1, \ldots 4,\ i = 1, 2, 3, \tag{8.7a}$$

$$P_1^4 = P_1^3\,, \quad P_2^4 = P_2^3\,, \tag{8.7b}$$

$$P_i^j = \frac{1}{w_i^\ell}\,(w_j^\ell + k_j^\ell - k_i^\ell)\,, \quad i \neq j\,, \quad i, j = 1, 2, 3, \tag{8.7c}$$

gives formulas (8.5) with the same denominator for all the τ_i, *i.e.*, we have

$$\sum_{k=1}^{3} H_1^k\,\alpha'_k + H_1^4 = \sum_{k=1}^{3} H_2^k\,\alpha'_k + H_2^4 = \sum_{k=1}^{3} H_3^k\,\alpha'_k + H_3^4\,. \tag{8.8}$$

More precisely, we obtain

$$H_i^k = w_k^\ell + k_k^\ell\,, \qquad H_i^4 = w_3^\ell + k_3^\ell\,, \tag{8.9}$$

for all $i, j = 1, 2, 3$, and

$$K_i^a = \frac{1}{w_i^\ell}\,L_i^a \tag{8.10}$$

for all $i = 1, 2, 3$ and $a = 1, \ldots 4$, where the L's are homogeneous polynomials of degrees 2 with respect to the coefficients w_i^Q, u_i^Q, v_i^ℓ and k_i^ℓ. The element P is not unique and we can obtain from other elements in $PGL(4, \mathbb{R})$ such a common denominator for the τ's.

Besides, with this admissible definition of P, we define the '*reduced time stamps*' τ_i^R such that

$$\tau_i^R \equiv \tau_i - \tau_i^{vp}\,, \tag{8.11}$$

where

$$\tau_i^{vp} = \left(\frac{\sum_{a=1}^4 K_i^a}{\sum_{b=1}^4 H_i^b} \right) \tag{8.12}$$

are the time stamps obtained when setting $\alpha_i' = 1$ $(i = 1, 2, 3)$. These reduced time stamps depend also on the angles α_k', but now, in addition, we make the following change of variables. We set

$$\alpha_i' = \mu_i + 1\,, \quad i = 1, 2, 3. \tag{8.13}$$

This change of variables is also a particular projective transformation (in addition to P) since it is an affine transformation. As a consequence, we obtain the following remarkable expressions for the reduced time stamps:

$$\tau_i^R = \left(\frac{\sum_{j=1}^3 A_i^j\, \mu_j}{\sum_{k=1}^3 B^k\, \mu_k + B^4} \right), \tag{8.14}$$

where the A_i^j are, respectively, fractions of homogeneous polynomial numerators of degrees 3 and homogeneous polynomial denominators of degrees 2 with respect to the variables u_i^Q, v_j^Q, w_k^ℓ and k_h^ℓ, and the B^a are homogeneous polynomials of degrees 1 with respect to the variables w_k^ℓ and k_h^ℓ only.

From all these preliminary results, we can deduce now the following list of remarks and consequences.

1. We shown that any projective transformation $P \in PGL(4, \mathbb{R})$ applied on the angles α_i is compatible with changes of projective frames on the celestial circles of the three events attached to the localized event e.

2. There always exists a projective transformation P equalizing the denominators of the relations (8.5) and such that the relations (8.5) expressed a projective transformation (**PT**) in $PGL(4, \mathbb{R})$ from the space of angles to the space of localized events. Then, there are two consequences:

    - The relations (8.5) with the denominators equalized are the defining relations of the ‘*soldering map*’ from the projective space $\mathbb{R}P^3$ of angles to the the manifold $\mathcal{M}$ of localized events. This soldering is a birational local map from $\mathbb{R}P^3$ to $\mathcal{M}$. It is only a local map because, from (8.5), if the angles tend to infinity then we pass from $\mathbb{R}P^3$ to the two-dimensional subspace $\mathbb{R}P^2$ meanwhile we get only one point for the corresponding limit event e.

- If $e^\star$ is another localized event attached to the events $E^\star$, $\widetilde{E}^\star$ and $\widehat{E}^\star$, then, it exists a **PT** from the coordinates $\tau_i^\star$ of $e^\star$ to the coordinates τ_i of e. Thus, $\mathcal{M}$ is a generalized Cartan space on $\mathbb{R}P^3$.

3. Let e and $e^\star$ be two localized events associated, respectively, to the unprimed angles α_i and $\alpha_i^\star$; and then, to the primed angles α_i' and $\alpha_i'^\star$ from the **PT** P and $P^\star$. Also, let μ_i and $\mu_i^\star$ be the translated angles such that $\alpha_i' = \mu_i + 1$ and $\alpha_i'^\star = \mu_i^\star + 1$. Now, if the $\mu_i^\star$ are obtained from the μ_i by a conformal transformation, then, it is easy to show that the angles $\alpha_i'^\star$ are deduced from the angles α_j' by a **PT**. Therefore, if the $\tau_i^\star$ associated to $e^\star$ are obtained from the τ_k associated to e by a **PT**, we deduce that the reduced time stamps $\tau_i^{R\star}$ are obtained from the reduced time stamps τ_j^R by a conformal transformation. The reciprocal is not true.

4. The acronym '*vp*' used as a superscript in the definition of the time stamps τ_j^{vp} means "*virtual point*" or "*vanishing point*" as well. Indeed, even if the point with coordinates $(\tau_1^{vp}, \tau_2^{vp}, \tau_3^{vp})$ can be positioned in the three-dimensional grid of the RPS, it does not correspond to any event since only one event is attached to the triplet of event E, $\widetilde{E}$ and $\widehat{E}$; so its virtuality. Additionally, it is the so-called vanishing point of the projective geometry well-known by painters drawing perspectives on their canvas; so the terminology. We suggest the existence of a possible "*Big-Bang effect*" due to this spacetime perspective relative to the vanishing points.

5. The **PT**s (8.5) with (8.8) can be recast within the framework of the Lie groupoid structures. For, we define, first, the "*data-point*" T_e to be the set of all of the time stamps collected at the events E, $\widetilde{E}$ and $\widehat{E}$ to localize e, and moreover, we denote by $\mathcal{T}$ the set of all such data-points T_e. We assume $\mathcal{T}$ to be locally a smooth manifold. Then, we shown that given two data-points T_E and $T_{e^\star}$, then, the 3-position $p_{e^\star}$ is obtained from

the 3-position p_e by a **PT** defined explicitly and univocally from T_e and $T_{e^\star}$. Hence, we can define the Lie groupoid $\mathcal{G} \rightrightarrows \mathcal{T}_s \times \mathcal{T}_t$ of **PT**s such that $\pi_s : \mathcal{G} \longrightarrow \mathcal{T}_s \equiv \mathcal{T}$ is the 'source map' and $\pi_t : \mathcal{G} \longrightarrow \mathcal{T}_t \equiv \mathcal{T}$ is the 'target map' of the groupoid. Then, the **PT**s deduced from any pair $(T_e, T_{e^\star}) \in \mathcal{T}_s \times \mathcal{T}_t$ define sections of $\mathcal{G}$. We can say that the translations from the source $T_e \in \mathcal{T}_s$ to the target $T_{e^\star} \in \mathcal{T}_t$ is in a one-to-one correspondence with a **PT** defining $p_{e\star}$ from p_e. In other words, the projective structure given by this set of **PT**s is not, a priori, strictly defined on $\mathcal{M}$ but rather on the data manifold $\mathcal{T}$. Nevertheless, to any data-point T_e corresponds a unique localized event e relative to the given RPS. The reciprocal is less obvious but it is also true. Indeed, e is the unique intersection point of three past null cones and only one triplet of such null cones have their apexes E, $\widetilde{E}$ and $\widehat{E}$ on the worldlines of the three emitters $\mathcal{E}$, $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$. Therefore, once the worldlines of $\mathcal{E}$, $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$, $\mathcal{S}$ are known from this given RPS, then, all the data need to localize e can be reached, and thus, T_e. Hence, we can say also that we have a Lie groupoid structure on $\mathcal{M}$ meaning that given p_e and $p_{e^\star}$ only we can deduce the unique **PT** compatible with the localization process to pass from p_e to $p_{e^\star}$. This **PT** is not applied on the whole of events in the grid. It is not a **PT** of the grid.

Also, we can say that a mere translation from p_e to $p_{e^\star}$ in the grid is, somehow, "converted" in a **PT** "compatible" with the localization process. By "compatible" we mean that the translations in the grid, for instance, cannot be directly physically observed contrary to the admissible **PT** on the celestial circles. And, moreover, assuming that we are not permanently drunk, lucidly looking at two simultaneous realities hierarchized according to our degree of consciousness into an "appearance" and a "reality," then, if we see only one "manifest image" [Ros12, Ros93b, vF99, Ros93a] on each celestial circle, then, this is just "the" reality... Thus, those transformations such as the translations or any transformation in the affine group must be interpreted or, somehow, "converted" into a manifest **PT**. But, we can avoid such conversion or interpretation considering that the grid has the structure of a projective space onto which transformations in the affine group, for instance, are forbidden, useless or not physical because physically not manifest.

From a more mathematical viewpoint, if, on the one hand, the (finite) local **PT**s are defined as elements of a Lie groupoid $\mathcal{G}$ over $\mathcal{M} \times \mathcal{M}$, then, on the other hand, from the present particular groupoid structure, the corresponding Lie algebroid is just identified with the module of vector fields on $\mathcal{M}$. In other words, the tensorial calculus must be a projective tensorial calculus over $\mathcal{M}$. As a consequence, the connections on $\mathcal{M}$ must be projective Cartan connections.

Moreover, the latter can be reduced to projective connections on each celestial circle in accordance with a mathematical procedure/computation analogous to the one giving the transformation formulas (8.6) on each celestial circle from the general transformation formulas (8.5) on the whole of $\mathcal{M}$.

Also, other reduced projective connections could be deduced and applied on the space of reduced time stamps. Hence, because the data space $\mathcal{T}$ is locally homeomorphic to $\mathcal{M}$ (we assume that it is, actually, diffeomorphic) we can make the geometrical computations on $\mathcal{M}$ in the abstract way, *i.e.*, avoiding to consider the full set of time stamps of T_e and considering only the restricted set of time stamps of p_e as much as only infinitesimal, tensorial computations are carried out; and thus, the origin of the "infinitesimal" projective geometry of $\mathcal{M}$ (but the finite projective geometry on $\mathcal{M} \times \mathcal{M}$ via the groupoid $\mathcal{G}$).

Lastly, we call '*anchoring worldline*' the worldline ${}^{\mathcal{S}}W$ of the emitter $\mathcal{S}$, and we call the '*anchor*' a of e the event $a \in {}^{\mathcal{S}}W$ such that the time stamp $\mathring{\tau}_{4,a}$ emitted by $\mathcal{S}$ at a is such that $\mathring{\tau}_{4,a} = f^{-}_{{}^{\mathcal{S}}W}(e)$.

C. The protocol of localization in a $(3+1)$-dimensional spacetime $\mathcal{M}$ modeled on $\mathbb{R}P^4$

The generalization of the previous protocol follows a similar process with five emitters $\mathcal{E}$, $\overline{\mathcal{E}}$, $\widetilde{\mathcal{E}}$, $\widehat{\mathcal{E}}$ and $\mathring{\mathcal{E}}$ broadcasting five time stamps, respectively, τ_1, τ_2, τ_3, τ_4 and $\mathring{\tau}_5$. They constitute five RPSs made, each, by four emitters among the five with the fifth one used for the localization of spacetime events denoted by e. Also, we denote, as in the precedent sections, by $\mathcal{U}$ the user and by $\mathcal{C}$, $\overline{\mathcal{C}}$, $\widetilde{\mathcal{C}}$, $\widehat{\mathcal{C}}$ and $\mathring{\mathcal{C}}$ the celestial spheres of the five emitters. The five grids of these five RPSs are Euclidean spaces $\mathbb{R}^4$. The passage from any grid to another one among the four others is a change of charts which is well-defined once the dated trajectories of the five emitters in the grids are obtained from each RPS and recorded.

For the sake of arguments, we consider only the RPS made with the first four emitters, namely, $\mathcal{E}$, $\overline{\mathcal{E}}$, $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ and its associated grid with the four time stamps τ_1, τ_2, τ_3 and τ_4 defining the so-called 4-positions of the events in this grid. Then, the fifth emitter $\mathring{\mathcal{E}} \equiv \mathcal{S}$ is used in complement for the localization process. Consequently, the worldline $\mathring{W}$ of $\mathring{\mathcal{E}}$ is the anchoring worldline of the relativistic location system.

Now, we consider only the set of particular events represented on the figures VIII.6, VIII.7 and VIII.8 with their corresponding tables of 4-positions.

The figure VIII.6 shows the different events, namely, E on the worldline W of $\mathcal{E}$, $\overline{E}$ on the worldline $\overline{W}$ of $\overline{\mathcal{E}}$, $\widetilde{E}$ on the worldline $\widetilde{W}$ of $\widetilde{\mathcal{E}}$ and $\widehat{E}$ on the worldline $\widehat{W}$ of $\widehat{\mathcal{E}}$ at which the event e manifests on their respective celestial spheres. We assume that the data of localization for e collected at the events E, $\overline{E}$, $\widetilde{E}$ and $\widehat{E}$ are sent to the user and they are received at the events, respectively, U, $\overline{U}$, $\widetilde{U}$ and $\widehat{U}$ on the worldline ${}^{\mathcal{U}}W$ of $\mathcal{U}$.

The figure VIII.7 indicates, first, the events $\overline{E}'$, $\widetilde{E}'$ and $\widehat{E}'$ from which the 4-position of the event E can be known in the grid (see Table I), and second, two other events, namely, $\mathring{E}'$ and e, which are observed on the celestial sphere $\mathcal{C}$ of the emitter $\mathcal{E}$ at E. Obviously, e is the event to be localized and $\mathring{E}'$ is a particular event on the worldline of $\mathring{\mathcal{E}}$ which broadcast the time stamp $\mathring{\tau}_5'$ to E used for the localization process. Similar figures could be indicated concerning the three other events $\overline{E}$, $\widetilde{E}$ and $\widehat{E}$ on the figure VIII.6 but there are not really necessary for the description of the localization process below. These unnecessary supplementary figures would indicate supplementary events on the worldline of $\mathring{\mathcal{E}}$ such as, for instance, $\mathring{E}^\bullet$ broadcasting (see Figure VIII.8) the time stamp $\mathring{\tau}_5^\bullet$ to the event $\overline{E}$ of the figure VIII.7. These particular time stamps are denoted by $\mathring{\tau}_5$ (with different superscripts) and they are sent from different events on the worldline of $\mathring{\mathcal{E}}$ to the other four emitters.

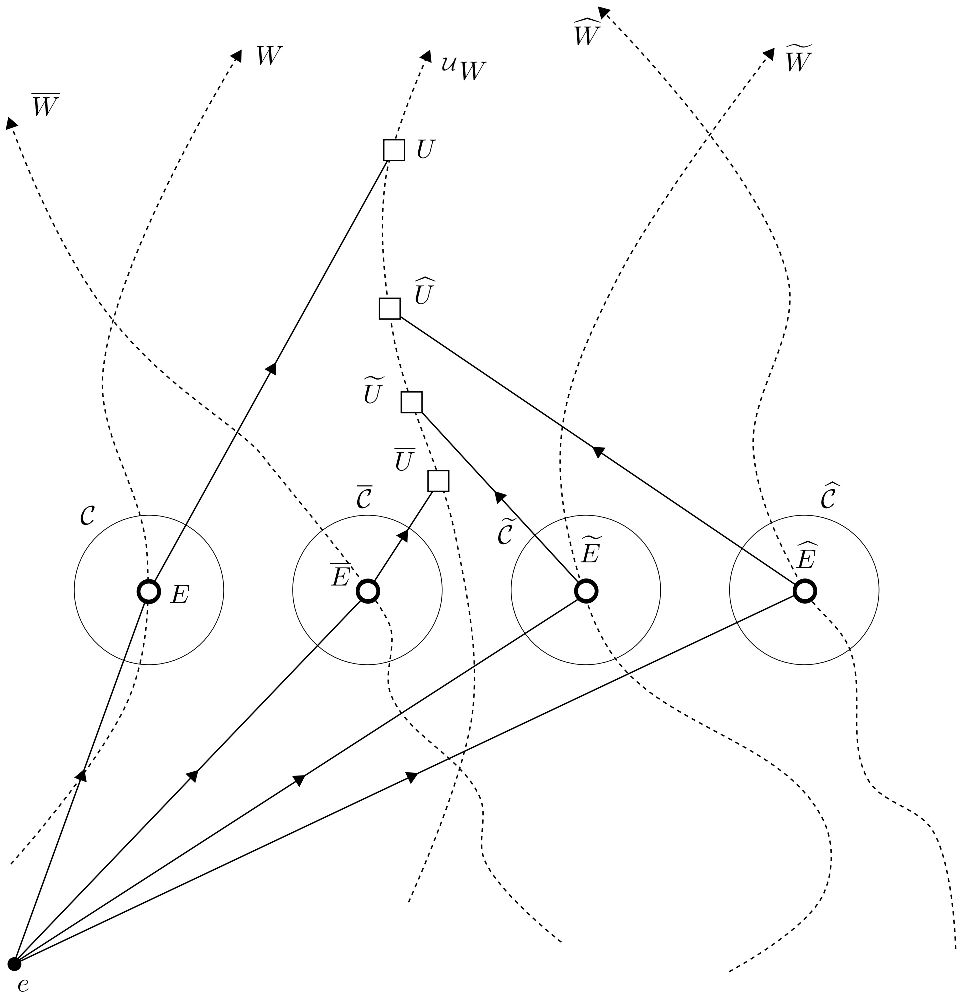


Figure VIII.6. The event e in the four past null cones of the four events E, $\overline{E}$, $\widetilde{E}$ and $\widehat{E}$. This event e is observed on their respective celestial spheres $\mathcal{C}$, $\overline{\mathcal{C}}$, $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$.

Also, angles are evaluated on each celestial sphere from optical devices and compasses providing pairs of angle, namely, (α, β) ascribed to each “bright point” observed and tracked on any given celestial sphere. Actually, each celestial sphere (homeomorphic to S^2) is considered as the union of a circle and two hemispheres. They are topological sets of which the first one is a closed set and also the common boundary of the seconds which are two open sets in S^2. In addition, each hemisphere is embedded in an open, connected and simply connected set in $\mathbb{R}P^2$ and, moreover, each hemisphere is supplied with a given projective frame made of four particular points to be specified in the sequel.

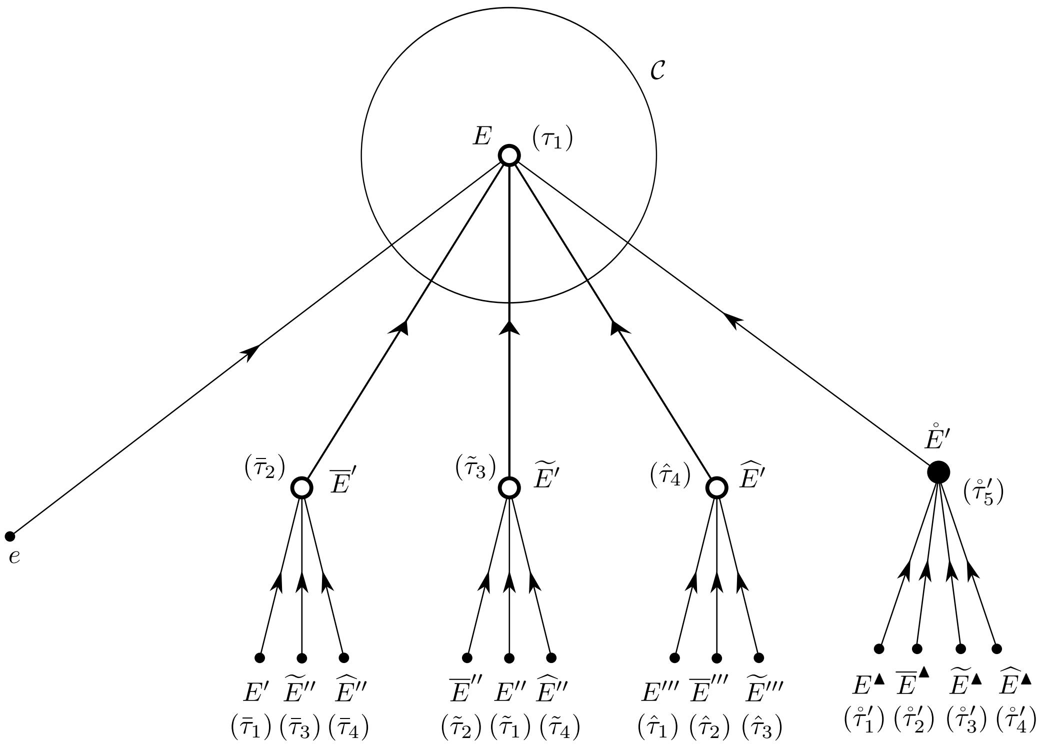


Figure VIII.7. The event E in the five future null cones of the five events e, $\overline{E}'$, $\widetilde{E}'$, $\widehat{E}'$ and $\mathring{E}'$.

Event	4-position
$\overline{E}'$	$(\bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3, \bar{\tau}_4)$
$\widetilde{E}'$	$(\tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_3, \tilde{\tau}_4)$
$\widehat{E}'$	$(\hat{\tau}_1, \hat{\tau}_2, \hat{\tau}_3, \hat{\tau}_4)$
E	$(\tau_1, \bar{\tau}_2, \tilde{\tau}_3, \hat{\tau}_4)$
$\mathring{E}'$	$(\mathring{\tau}'_1, \mathring{\tau}'_2, \mathring{\tau}'_3, \mathring{\tau}'_4)$

Table I. The 4-positions of the events on the Figure VIII.7.

One hemisphere is made of a little spherical cap, as little as possible, and the other is its complementary hemisphere in S^2 with their common boundary to be, for instance, a polar circle. This choice is motivated from metrological considerations. Indeed, we want the probability of passage from one hemisphere to the other to be as lower as possible when tracking trajectories

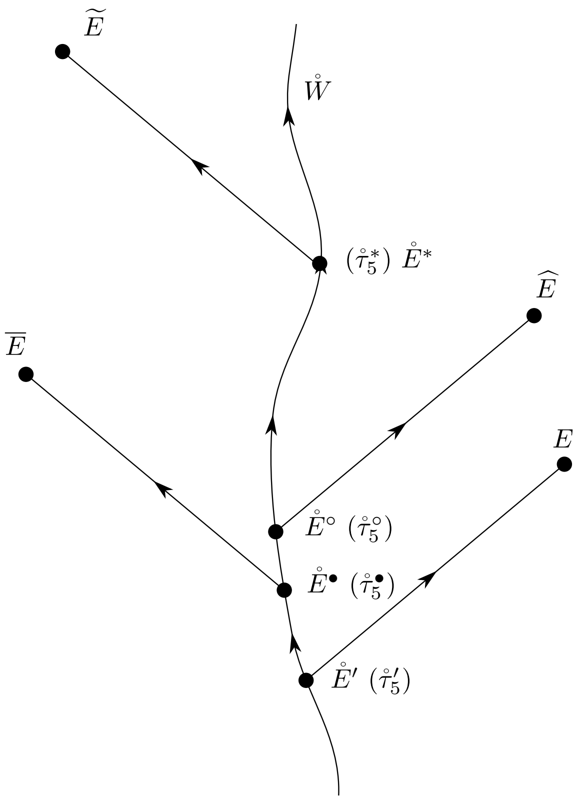


Figure VIII.8. An example of successive events $\mathring{E}'$, $\mathring{E}^\bullet$, $\mathring{E}^\circ$ and $\mathring{E}^*$ on the anchoring worldline of $\mathring{\mathcal{E}}$ broadcasting their coordinates $\mathring{\tau}_5$ towards the four events E, $\overline{E}$, $\widehat{E}$ and $\widetilde{E}$.

of moving points on the celestial spheres. Nevertheless, we provide each celestial sphere with a computing device insuring, on the polar circle, the change of projective frame from one hemisphere to the other and, for each moving point, recording the signature of its passage, *viz*, a plus or minus mark. As a consequence, we can track more completely moving "bright points," and then, we can position these points in only one specified, given system of projective coordinates common to the two hemispheres minus a point (the north pole for instance) to which is ascribed an identifying symbol instead of two angles. Then, we can establish the correspondences between the pairs of angles in the two hemispheres and on the polar circle.

We represent usually one hemisphere embedded in $\mathbb{R}P^2$ by a two-dimensional disk in $\mathbb{R}^2$ added with a half of the polar circle. Then, we have projective frames made of four projective points $[\infty, 0]$, $[0, \infty]$, $[0, 0]$ and $[1, 1]$ with the first two on the polar circle (see Figure

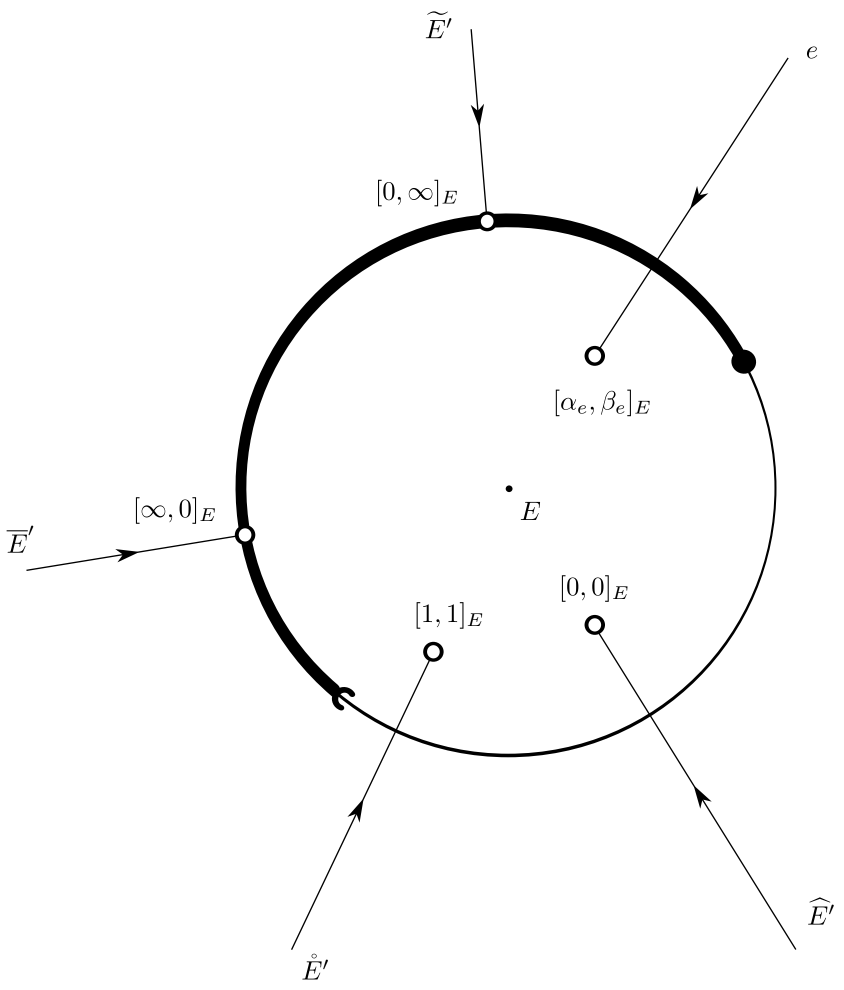


Figure VIII.9. The projective disk at the event E associated to the celestial sphere $\mathcal{C}$ of the emitter $\mathcal{E}$.

VIII.9). Also, a projective point $[\alpha_e, \beta_e]$ is ascribed to the event e observed on the celestial spheres. More precisely, one of the two projective spaces $\mathbb{R}P^2$ attached to the celestial sphere $\mathcal{C}$ of $\mathcal{E}$ at the event E is represented in the Figure VIII.9. Also, a first projective frame $\mathfrak{F}_E \equiv \{[\infty,0]_E, [0,\infty]_E, [0,0]_E, [1,1]_E\}$ attached to this projective space is represented providing the projective coordinates $[\alpha,\beta]_E$. Also, a second projective frame $\mathfrak{F}'_E \equiv \{[\infty,0]'_E, [0,\infty]'_E, [0,0]'_E, [1,1]'_E\}$ is defined from a change of projective frame from $\mathfrak{F}_E$ to $\mathfrak{F}'_E$. This change of frame is based on pairs of numerical values given, for instance, by the first family of time stamps, namely, (τ_1, τ_2) obtained from the first emitters $\mathcal{E}$ and $\overline{\mathcal{E}}$.

More precisely, we define the first four following correspondences:

$$\begin{array}{ccccc} e & \longleftrightarrow & [\alpha_e,\beta_e]_E & \longleftrightarrow & [\tau_e^E,\bar{\tau}_e^E]'_E\,, \\ \overline{E}' & \longleftrightarrow & [\infty,0]_E & \longleftrightarrow & [\bar{\tau}_1,\bar{\tau}_2]'_E\,, \\ \widetilde{E}' & \longleftrightarrow & [0,\infty]_E & \longleftrightarrow & [\tilde{\tau}_1,\tilde{\tau}_2]'_E\,, \\ \widehat{E}' & \longleftrightarrow & [0,0]_E & \longleftrightarrow & [\hat{\tau}_1,\hat{\tau}_2]'_E\,, \end{array}$$

but with the additional correspondence

$$\mathring{E}' \longleftrightarrow [1,1]_E \longleftrightarrow [\mathring{\tau}'_5,\lambda]'_E\,,$$

where λ is a time value free a varying at this step of the process. Also, it is important to note that $\mathring{\tau}'_5$ can be one of the four other time stamps received at $\mathring{E}'$ by $\mathring{\mathcal{E}}$ from the four other satellites, *i.e.*, it can be equal to $\mathring{\tau}'_1$, $\mathring{\tau}'_2$, $\mathring{\tau}'_3$ or $\mathring{\tau}'_4$. But, these four values are clearly independent on the whole of the other time stamps such as, for instance, τ_1, $\hat{\tau}_3$, $\tilde{\tau}_4$, etc., involved in the localization process, all the more so as that these $\mathring{\tau}'_i$'s depend on the worldline of $\mathring{\mathcal{E}}$. Hence, $\mathring{\tau}'_5$ is considered as an independent time variable in the process; so a fifth supplementary time stamp indexed by the number 5. In addition, the parameter λ is, actually, well-defined, as shown in the sequel, from the complete description of the process of localization.

Furthermore, we can have the table II of attributions based on the following families of two time stamps: τ_1 and τ_2 for E, τ_2 and τ_3 for $\overline{E}$, τ_3 and τ_4 for $\widetilde{E}$, and τ_4 and τ_1 for $\widehat{E}$ (only the correspondences $[\,.\,,\,.\,]_E \longleftrightarrow [\,.\,,\,.\,]_E$ are indicated in this table; the others are not need for the explanations given below and they are indicated by the marks '$***$'):

Then, we determine the change of projective frame in $\mathbb{R}P^2$ on the celestial sphere $\mathcal{C}$ of $\mathcal{E}$ at E. For, we must compute the matrix K such as

$$K = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & k \end{pmatrix} \tag{8.15}$$

associated with this change of frame. This matrix K is defined from the following correspon-

Table II. Attributions of time stamps, angles and events.

	$\mathcal{E}$	$\overline{\mathcal{E}}$	$\widetilde{\mathcal{E}}$	$\widehat{\mathcal{E}}$	$\mathring{\mathcal{E}}$	event	family of time stamps
E	—	$\overline{E}'$	$\widetilde{E}'$	$\widehat{E}'$	$\mathring{E}'$	e	(τ_1, τ_2)
	—	$[\infty, 0]_E$	$[0, \infty]_E$	$[0, 0]_E$	$[1, 1]_E$	$[\alpha_e, \beta_e]_E$	
	—	$[\bar\tau_1, \bar\tau_2]'_E$	$[\tilde\tau_1, \tilde\tau_2]'_E$	$[\hat\tau_1, \hat\tau_2]'_E$	$[\mathring{\tau}'_5, \lambda]'_E$	$[\tau_e^E, \bar\tau_e^E]'_E$	
$\overline{E}$	$E^\bullet$	—	$\widetilde{E}^\bullet$	$\widehat{E}^\bullet$	$\mathring{E}^\bullet$	e	(τ_2, τ_3)
	$[\infty, 0]_{\overline{E}}$	—	$[0, 0]_{\overline{E}}$	$[0, \infty]_{\overline{E}}$	$[1, 1]_{\overline{E}}$	$[\bar\alpha_e, \bar\beta_e]_{\overline{E}}$	
	$***$	—	$***$	$***$	$[\mathring{\tau}^\bullet_5, \bar\lambda]'_{\overline{E}}$	$[\bar\tau_e^{\overline{E}}, \tilde\tau_e^{\overline{E}}]'_{\overline{E}}$	
$\widetilde{E}$	E^*	$\overline{E}^*$	—	$\widehat{E}^*$	$\mathring{E}^*$	e	(τ_3, τ_4)
	$[0, \infty]_{\widetilde{E}}$	$[0, 0]_{\widetilde{E}}$	—	$[\infty, 0]_{\widetilde{E}}$	$[1, 1]_{\widetilde{E}}$	$[\tilde\alpha_e, \tilde\beta_e]_{\widetilde{E}}$	
	$***$	$***$	—	$***$	$[\mathring{\tau}^*_5, \tilde\lambda]'_{\widetilde{E}}$	$[\tilde\tau_e^{\widetilde{E}}, \hat\tau_e^{\widetilde{E}}]'_{\widetilde{E}}$	
$\widehat{E}$	E°	$\overline{E}^\circ$	$\widetilde{E}^\circ$	—	$\mathring{E}^\circ$	e	(τ_4, τ_1)
	$[0, 0]_{\widehat{E}}$	$[0, \infty]_{\widehat{E}}$	$[\infty, 0]_{\widehat{E}}$	—	$[1, 1]_{\widehat{E}}$	$[\hat\alpha_e, \hat\beta_e]_{\widehat{E}}$	
	$***$	$***$	$***$	—	$[\mathring{\tau}^\circ_5, \hat\lambda]'_{\widehat{E}}$	$[\hat\tau_e^{\widehat{E}}, \tau_e^{\widehat{E}}]'_{\widehat{E}}$	

dences in $\mathbb{R}^3$:

$$\overline{E}' \ : [\infty, 0]_E \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \xrightarrow{K} \qquad [\bar\tau_1, \bar\tau_2]'_E \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{where} \quad \begin{cases} \bar\tau_1 & = a/c \\ \bar\tau_2 & = b/c \end{cases}$$

$$\widetilde{E}' \ : [0, \infty]_E \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \xrightarrow{K} \qquad [\tilde\tau_1, \tilde\tau_2]'_E \equiv \begin{pmatrix} d \\ e \\ f \end{pmatrix} \quad \text{where} \quad \begin{cases} \tilde\tau_1 & = d/f \\ \tilde\tau_2 & = e/f \end{cases}$$

$$\widehat{E}' \ : [0, 0]_E \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \xrightarrow{K} \qquad [\hat\tau_1, \hat\tau_2]'_E \equiv \begin{pmatrix} g \\ h \\ k \end{pmatrix} \quad \text{where} \quad \begin{cases} \hat\tau_1 & = g/k \\ \hat\tau_2 & = h/k \end{cases}$$

$$\mathring{E}' \,:\, [1,1]_E \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\quad K \quad} [\mathring{\tau}'_5, \lambda]'_E \equiv \begin{pmatrix} a+d+g \\ b+e+h \\ c+f+k \end{pmatrix} \text{ where } \begin{cases} \mathring{\tau}'_5 & = \left(\frac{a+d+g}{c+f+k}\right) \\ \lambda & = \left(\frac{b+e+h}{c+f+k}\right) \end{cases}$$

$$e \,:\, [\alpha_e, \beta_e]_E \equiv \begin{pmatrix} \alpha_e \\ \beta_e \\ 1 \end{pmatrix} \xrightarrow{\quad K \quad} [\tau_e^E, \bar{\tau}_e^E]'_E \equiv \begin{pmatrix} u \\ v \\ w \end{pmatrix} \text{ where } \begin{cases} \tau_e^E & = u/w \\ \bar{\tau}_e^E & = v/w \end{cases}$$

and

$$\begin{aligned} u &= \alpha_e\, a + \beta_e\, d + g\,, \\ v &= \alpha_e\, b + \beta_e\, e + h\,, \\ w &= \alpha_e\, c + \beta_e\, f + k\,. \end{aligned}$$

From, we deduce the four following linear equations:

$$\begin{cases} (\bar{\tau}_1 - \mathring{\tau}'_5)\, x + (\tilde{\tau}_1 - \tau'_5)\, y + (\hat{\tau}_1 - \tau'_5) = 0\,, \\ (\bar{\tau}_2 - \lambda)\, x + (\tilde{\tau}_2 - \lambda)\, y + (\hat{\tau}_2 - \lambda) = 0\,, \end{cases} \tag{8.16a}$$

$$\begin{cases} \alpha_e\, (\bar{\tau}_1 - \tau_e^E)\, x + \beta_e\, (\tilde{\tau}_1 - \tau_e^E)\, y + (\hat{\tau}_1 - \tau_e^E) = 0\,, \\ \alpha_e\, (\bar{\tau}_2 - \bar{\tau}_e^E)\, x + \beta_e\, (\tilde{\tau}_2 - \bar{\tau}_e^E)\, y + (\hat{\tau}_2 - \bar{\tau}_e^E) = 0\,, \end{cases} \tag{8.16b}$$

where $x \equiv c/k$ and $y \equiv f/k$, and where x, y, λ, τ_e^E and $\bar{\tau}_e^E$ are the unknowns. From the system (8.16a), we obtain, first, the values for x and y, and second, from (8.16b), we obtain τ_e^E and $\bar{\tau}_e^E$ such that

$$\tau_e^E = \frac{P(\lambda, \mathring{\tau}'_5, \alpha_e, \beta_e)}{P_0(\lambda, \mathring{\tau}'_5, \alpha_e, \beta_e)}\,, \tag{8.17a}$$

$$\bar{\tau}_e^E = \frac{\overline{P}(\lambda, \mathring{\tau}'_5, \alpha_e, \beta_e)}{P_0(\lambda, \mathring{\tau}'_5, \alpha_e, \beta_e)}\,, \tag{8.17b}$$

where P, $\overline{P}$ and P_0 are polynomials of degrees one with respect to λ and $\mathring{\tau}'_5$ with coefficients as polynomials of degrees one with respect to α_e and β_e.

We compute also the four other pairs of time stamps ascribed to the event e, *i.e.*, $(\bar{\tau}_e^{\overline{E}}, \tilde{\tau}_e^{\overline{E}})$, $(\tilde{\tau}_e^{\widetilde{E}}, \hat{\tau}_e^{\widetilde{E}})$ and $(\hat{\tau}_e^{\widehat{E}}, \tau_e^{\widehat{E}})$ (see Table II) obtained at the events, respectively, $\overline{E}$, $\widetilde{E}$ and $\widehat{E}$. We obtain expressions similar to the expressions (8.17) with respect to the other λ's, τ_5's, α's and β's. And then, we set the following constraints:

$$\begin{cases} \tau_e^E &= \tau_e^{\widehat{E}}\,, \\ \bar\tau_e^E &= \bar\tau_e^{\overline{E}}\,, \\ \tilde\tau_e^{\overline{E}} &= \tilde\tau_e^{\widetilde{E}}\,, \\ \hat\tau_e^{\widetilde{E}} &= \hat\tau_e^{\widehat{E}}\,. \end{cases} \tag{8.18}$$

These constraints are well-justified since any e has only one 4-position. Then, we deduce four equations of the form

$$\lambda_1 = \left(\frac{u\,\lambda_2 + w}{w\,\lambda_2 + r}\right), \tag{8.19}$$

for any pair (λ_1, λ_2) of distinct λ in the set $\{\lambda, \bar\lambda, \tilde\lambda, \hat\lambda\}$ from which we deduce one quadratic equation for each λ with coefficients independent on the other λ's but, nevertheless, depending on the angles and the various time stamps τ. Therefore, we have proved that each λ has a value which is independent on the other λ's. But, in addition, the λ's must also be independent on the angles because there are ascribed to the projective points $[1,1]$ independent on the events such as e. Hence, we can arbitrary fix the values for the λ's. The natural choice is to set the following:

$$\lambda \equiv \mathring\tau_5'\,, \qquad \bar\lambda \equiv \mathring\tau_5^\bullet\,, \qquad \tilde\lambda \equiv \mathring\tau_5^*\,, \qquad \hat\lambda \equiv \mathring\tau_5^\circ\,. \tag{8.20}$$

In return, from (8.19) with (8.20), we deduce also four fractional relations between, on the one hand, the α's, and, on the other hand, the β's. The general form of these relations is the following, for instance, for β_e:

$$\beta_e = \left(\frac{u\,\alpha_e + \bar u\,\bar\alpha_e + \tilde u\,\tilde\alpha_e + \hat u\,\hat\alpha_e + r}{w\,\alpha_e + \bar w\,\bar\alpha_e + \tilde w\,\tilde\alpha_e + \hat w\,\hat\alpha_e + s}\right), \tag{8.21}$$

where the coefficients u, $\bar u$, etc. are depending on the times stamps except those ascribed to the localized event.

Then, we obtain the 4-position $p_e \equiv (\tau_e, \bar\tau_e, \tilde\tau_e, \hat\tau_e)$ for e in the grid such that $\tau_e \equiv \tau_e^E$, $\bar\tau_e \equiv \bar\tau_e^E$, $\tilde\tau_e \equiv \tilde\tau_e^{\widetilde{E}}$ and $\hat\tau_e \equiv \hat\tau_e^{\widetilde{E}}$ depending on the four angles α_e, $\bar\alpha_e$, $\tilde\alpha_e$ and $\hat\alpha_e$ only and the time stamps. For instance, τ_e is such that

$$\tau_e = \left(\frac{p\,\alpha_e + \bar{p}\,\bar{\alpha}_e + \tilde{p}\,\tilde{\alpha}_e + \hat{p}\,\hat{\alpha}_e + q}{m\,\alpha_e + \bar{m}\,\bar{\alpha}_e + \tilde{m}\,\tilde{\alpha}_e + \hat{m}\,\hat{\alpha}_e + n} \right). \tag{8.22}$$

As a result, from 1) the form of this expression which is the same for each time stamp of the 4-position of e, and 2) following the same reasoning as in the precedent section for a $(2+1)$-dimensional spacetime, the group $PGL(5,\mathbb{R})$ acts on $\mathcal{M}$ via a projective transformation applied to the four angles α_e, $\bar{\alpha}_e$, $\tilde{\alpha}_e$ and $\hat{\alpha}_e$.

Then, we can almost completely paraphrase what we described from the page 123 in the precedent section adding just one time stamp $\bar{\tau}$ and another supplementary angle $\bar{\alpha}$. And then, following the same reasoning, we deduce that $\mathcal{M}$ is modeled on $\mathbb{R}P^4$ and that it is embedded in $\mathbb{R}^5$. Finally, we denote by τ_5 the fifth coordinate of the fibers of the submersion $\mathbb{R}^5$ to $\mathcal{M}$. This supplementary time stamp τ_5 differing from the time stamp $\mathring{\tau}_5$ broadcast by $\mathring{\mathcal{E}}$ is, actually, defined from a particular 1-form and the anchoring worldline as we shall see in the net section.

IX. THE GEOMETRY OF $\mathcal{M}$ MODELED ON $\mathbb{R}P^4$

From now, we consider that the spacetime manifold $\mathcal{M}$ is locally homeomorphic to $P\mathbb{R}^4$ and that the latter is deduced from a particular manifold denoted by $\widehat{\mathcal{M}}$ such that $\dim\widehat{\mathcal{M}} = 5$. More precisely, $\mathcal{M}$ is identified with a particular leaf of codimension one of a five-dimensional foliated manifold $\widehat{\mathcal{M}}$. Thus, we need to introduce a fifth time stamp τ_5 of which the definition will be clarified from the extension of the metric field of $\mathcal{M}$ in $\widehat{\mathcal{M}}$. The way we embed $\mathcal{M}$ into $\widehat{\mathcal{M}}$ is guided by this metric extension following the fundamental principle that the projective subvarieties defined by the metric g on each tangent space $T_e\mathcal{M}$ must also be projective subvarieties on the corresponding tangent space $T_{\hat{e}}\widehat{\mathcal{M}}$ defined by the extended metric $\hat{g}$. Then, to the S^3 spheres of intersection of null cones with three-dimensional space-like affine spaces in $T\mathcal{M}$, it must correspond S^4 spheres of intersection of null cones with four-dimensional space-like affine spaces in $T\widehat{\mathcal{M}}$. In particular, this involves that to any light-like vector field on $\mathcal{M}$ must correspond a unique light-like vector field on $\widehat{\mathcal{M}}$. This general principle of extension is historically due to A. Cayley. Also, this metric extension exhibits a particular 1-form on $\widehat{\mathcal{M}}$ which is ascribed to the Yano-Ishihara projecting 1-form of the foliation from which the projective structure of $\mathcal{M}$ is then made explicit as well as the fifth time stamp. Then, the first step is to obtain the extended metric field on $\widehat{\mathcal{M}}$ from the one defined on $\mathcal{M}$. As we shall see $\widehat{\mathcal{M}}$ is the "energyspacetime" where the concept of energy is attributed to a new geometrical dimension associated with τ_5 in addition to those of space and time. Also, we mention that there exists a lot of recent references on the projective geometry and its relation with the relativity [Sin56, Mik80, Mik96, Mat04, GN06, HL07, HL08, EM08, HL09b, HL09a, BDE09, CKM10, Sta11, Nur12, Kam12, Nur12, Hal12, GM, Mat12, vGM13].

A. The metric field defined on $\widehat{\mathcal{M}}$

We recall that the metric g on $\mathcal{M}$ is of the form $g = -\sum_{\alpha<\beta=1}^{4} \nu_\alpha \nu_\beta \, d\tau_\alpha \odot d\tau_\beta = -\sum_{\alpha<\beta=1}^{4} \sigma_\alpha \odot \sigma_\beta$ where the ν_σ's are four positive functions defined on $\mathcal{M}$ and depending on four parameters τ_α; and, moreover, $\sigma_\alpha \equiv \nu_\alpha \, d\tau_\alpha$. It can be shown that the signature of g is $(+,+,+,-)$.

From now, we consider that the four functions ν_α we denote from now by $\hat{\nu}_\alpha$ depend on the five parameters $\tau_{\mathfrak{s}}$ for $\mathfrak{s} = 1, \ldots, 5$ with τ_5 clearly specified further, and then, German letters $\mathfrak{h}$, $\mathfrak{k}$, $\mathfrak{s}$, ...etc., are used for indices from 1 to 5. We denote also by $\tilde{g}$ the metric substituted to g and such that

$$\tilde{g} \equiv e^{-2\varphi}\, g = -e^{-2\varphi} \sum_{\alpha<\beta=1}^{4} \hat{\nu}_\alpha \hat{\nu}_\beta\, d\tau_\alpha \odot d\tau_\beta = -e^{-2\varphi} \sum_{\alpha<\beta=1}^{4} \hat{\sigma}_\alpha \odot \hat{\sigma}_\beta \tag{9.1}$$

where φ is any smooth function depending on the five time stamps, and thus, defined on $\widehat{\mathcal{M}}$. In the Euclidean space $\mathbb{R}^5$ of points $\tau \equiv (\tau_1, \ldots, \tau_5)$, we want a metric $\hat{g}$ defined 1) from the $\hat{\sigma}_\alpha = \hat{\nu}_\alpha\, d\tau_\alpha$ and φ only, if possible, and such that 2) it is symmetric under the group of permutation S_4 exchanging the four 1-forms $\hat{\sigma}_\alpha$, 3) to any light-like 4-vector k with respect to g, and therefore to $\tilde{g}$ also, corresponds a unique light-like 5-vector $\hat{k}$ with respect to $\hat{g}$, and 4) its restriction on $\mathcal{M}$ is $\tilde{g}$. In other words, we seek for a metric $\hat{g}$ on $\widehat{\mathcal{M}}$ in a one-to-one correspondence with the metric $\tilde{g}$ on $\mathcal{M}$ parameterized by τ_5. Then, $\hat{g}$ is necessarily of the following form:

$$\hat{g} = \tilde{g} + a\, d\tau_5 \otimes d\tau_5 + \sum_{\alpha,\,\beta=1}^{4} c_{\alpha\beta}\, \hat{\sigma}_\alpha \odot \hat{\sigma}_\beta + d\tau_5 \odot \left(\sum_{\alpha=1}^{4} b_\alpha\, \hat{\sigma}_\alpha \right), \tag{9.2}$$

where a, b_α and $c_{\alpha\beta} = c_{\beta\alpha}$ are functions of the five coordinates $\tau_{\mathfrak{s}}$.[40] Then, let k be a light-like 4-vector on $\mathcal{M}$, *i.e.*, a 4-vector such that $\tilde{g}(k,k) = 0$, *i.e.*, $g(k,k) = 0$. We would like to define $\hat{g}$ and a light-like 5-vector $\hat{k} \equiv (k, k^5)$ such that $\hat{g}(\hat{k}, \hat{k}) = 0$. From the latter, we deduce that k^5, a, b_α and $c_{\alpha\beta}$ are such that $A\,(k^5)^2 + B\,k^5 + C = 0$, where $A \equiv a$, $B \equiv \sum_{\alpha=1}^{4} b_\alpha\, \hat{\nu}_\alpha\, k^\alpha$ and $C \equiv \sum_{\alpha,\beta=1}^{4} c_{\alpha\beta}\, \hat{\nu}_\alpha\, \hat{\nu}_\beta\, k^\alpha\, k^\beta$. The value k^5 exists if and only if $B^2 - 4\,A\,C \geq 0$, *i.e.*, $\sum_{\alpha=1}^{4}(b_\alpha^2 - 4\,a\,c_{\alpha\alpha})(\hat{\nu}_\alpha\, k^\alpha)^2 + 2\sum_{\alpha<\beta=1}^{4}(b_\alpha\, b_\beta - 4\,a\,c_{\alpha\beta})\,\hat{\nu}_\alpha\,\hat{\nu}_\beta\, k^\alpha\, k^\beta \geq 0$. Hence, if the relations

$$b_\alpha\, b_\beta - 4\,a\,c_{\alpha\beta} = \sqrt{(b_\alpha^2 - 4\,a\,c_{\alpha\alpha})(b_\beta^2 - 4\,a\,c_{\beta\beta})} + u\,(\delta_{\alpha\beta} - 1)\,, \tag{9.3}$$

hold where u is any function for taking account of $\tilde{g}(k,k) = 0$, *i.e.*, $\sum_{\alpha<\beta=1}^{4} \hat{\nu}_\alpha\, \hat{\nu}_\beta\, k^\alpha\, k^\beta = 0$, then, $B^2 - 4\,A\,C$ is a square. But, if we set

$$b_\alpha^2 = 4\,a\,c_{\alpha\alpha}\,, \tag{9.4}$$

[40] For a general approach, see: Eastwood and Gover [ER11].

then, from (9.3), we deduce that

$$b_\alpha\, b_\beta - 4\, a\, c_{\alpha\beta} = u\,(\delta_{\alpha\beta} - 1)\,, \tag{9.5}$$

and, in particular, if $\alpha = \beta$ we obtain the relation (9.4) compatible with the relation (9.5) which is thus admissible as a solution for all α and β. But, moreover, $\hat{k}$ is in a one-to-one correspondence with k assuming $k^5 \geq 0$ since we have also $B^2 - 4\,A\,C = 0$ whatever is the function u. Additionally, we consider $a \neq 0$ because we want $d\tau_5$ to appear in the definition of $\hat{g}$. Next, we change the notations. We set: $a \equiv \epsilon\,\phi^2 \neq 0$, $b_\alpha \equiv 2\,\phi\,\psi_\alpha$ where $\epsilon = \pm 1$. Then, from (9.5) we obtain

$$\hat{g} = \tilde{g} + \epsilon\,\phi^2\, d\tau_5 \otimes d\tau_5 + \epsilon \sum_{\alpha,\beta=1}^{4} \psi_\alpha\,\psi_\beta\,\hat{\sigma}_\alpha \odot \hat{\sigma}_\beta + 2\,\phi\, d\tau_5 \odot \left(\sum_{\beta=1}^{4} \psi_\beta\,\hat{\sigma}_\beta \right) + w \sum_{\alpha<\beta=1}^{4} \hat{\sigma}_\alpha \odot \hat{\sigma}_\beta\,, \tag{9.6}$$

where w comes from the function u. But, we want the signature of $\hat{g}$ to be $(+,+,+,+,-)$ and, in addition, the last term with w can be included in $\tilde{g}$. Therefore, we set $\epsilon = 1$ and $w \equiv 0$ and we obtain the metric $\hat{g}$ such that

$$\hat{g} = \tilde{g} + \phi^2\, d\tau_5 \otimes d\tau_5 + \sum_{\alpha,\beta=1}^{4} \psi_\alpha\,\psi_\beta\,\hat{\sigma}_\alpha \odot \hat{\sigma}_\beta + 2\,\phi\, d\tau_5 \odot \left(\sum_{\beta=1}^{4} \psi_\beta\,\hat{\sigma}_\beta \right). \tag{9.7}$$

Actually, $\hat{g}$ can be put in the following form:

$$\hat{g} = \tilde{g} + \hat{h}\,, \tag{9.8}$$

where

$$\hat{h} \equiv \left(\phi\, d\tau_5 + \textstyle\sum_{\alpha=1}^{4} \psi_\alpha\,\hat{\sigma}_\alpha\right) \otimes \left(\phi\, d\tau_5 + \textstyle\sum_{\alpha=1}^{4} \psi_\alpha\,\hat{\sigma}_\alpha\right), \tag{9.9}$$

and where ϕ and ψ_α are functions of the five coordinates $\tau_{\mathbf{5}}$.

Thus, we deduce the following:

Property 5. *Let $\widehat{\mathcal{H}}_\tau$ be the set of 5-vectors $\hat{k} \equiv (k, k^5)$ at $\tau \in \widehat{\mathcal{M}}$ such that $\hat{h}(\hat{k}, \hat{k}) = 0$ and $k^5 > 0$. Then, if $\hat{k} \in \widehat{\mathcal{H}}_\tau$, $\phi \neq 0$, $\psi_\alpha \neq 0$ for at least one α in $\hat{h}$ and*

$$\hat{g} = \tilde{g} + \hat{h}\,, \tag{9.10}$$

then, $\hat{k} \equiv (k, k^5)$ has the same type as k and, moreover, k^5 is unique for any given 4-vector k.

Remark 15. *In this property, the condition $\psi_\alpha \neq 0$ is essential since, on the contrary, we need to set $k^5 \equiv 0$ for any given k of the same type as $\hat{k}$. This would not be the required result for an extension in higher dimension. Additionally, and as a result, we would have $\hat{k} \equiv (k, 0)$ and the projective geometry would be reduced globally from $\mathbb{R}P^4$ to $\mathbb{R}P^3$ only. Hence, we would recover the situation presented in the precedent sections. In the present case with $\psi_\alpha \neq 0$, this reduction to $\mathbb{R}P^3$ could be realized, obviously, punctually at some isolated points in $\widehat{\mathcal{M}}$ where all of the ψ_α vanish.*

From now and throughout, we can set $\phi \equiv 1$ in $\hat{h}$. And then, we have the following definition.

Definition 18. *We denote by $\widehat{\mathcal{M}}_{\mathcal{R}} = (\widehat{\mathcal{M}}, \hat{g})$ the five dimensional Riemannian manifold defined by $\widehat{\mathcal{M}}$ and $\hat{g}$ such that*

$$\hat{g} = \tilde{g} + \hat{h}\,, \tag{9.11a}$$

$$\hat{h} = \hat{\sigma}_5 \otimes \hat{\sigma}_5\,, \tag{9.11b}$$

$$\hat{\sigma}_5 = d\tau_5 + \textstyle\sum_{\alpha=1}^{4} \psi_\alpha\, \hat{\sigma}_\alpha\,, \tag{9.11c}$$

$$d\hat{\sigma}_5 \neq 0 \tag{9.11d}$$

where $\psi_\alpha \in \mathcal{O}_{\widehat{\mathcal{M}}}$ are not all vanishing smooth functions on $\widehat{\mathcal{M}}$.

In this definition, $\hat{\sigma}_5$ must not be closed because if $d\hat{\sigma}_5 = 0$, then there exists a parameter $\hat{\lambda}$ such that locally $\hat{g} = \tilde{g} + d\hat{\lambda} \otimes d\hat{\lambda}$, and thus, we are in a situation equivalent to the case $\psi_\alpha = 0$ for all the indices α up to a linear change of variable on the fifth coordinate k^5.

We deduce also:

Property 6. *$\widehat{\mathcal{H}}$ is the set of 5-vectors $\hat{k} \equiv (k, k^5)$ such that $k^5 > 0$ and $\hat{\sigma}_5(\hat{k}) = 0$.*

Now, we would like also to complete the definition of $\hat{g}$, and thus, to find particular functions ψ_α. For, we denote by $\mathcal{S}$ the Pfaffian system of 1-forms such that $\mathcal{S} = \{\hat{\sigma}_1, \ldots, \hat{\sigma}_4\}$. As a four dimensional Pfaffian system in a five dimensional space, $\mathcal{S}$ is necessarily integrable. It can be shown also from the following simple computation: $d\hat{\sigma}_\alpha = d\big(\ln|\hat{\nu}_\alpha|\big) \wedge \hat{\sigma}_\alpha$ and then, in particular, $\hat{\sigma}_\alpha \wedge d\hat{\sigma}_\alpha = 0$. But, this condition is satisfied whatever are the functions ψ_α which remain, somehow, undetermined by the integrability of $\mathcal{S}$. Moreover, we want to preserve the S_4 invariance which is at the heart of the determination of $\hat{g}$.

Then, let $\widehat{\mathcal{S}}$ be the Pfaffian system such that $\widehat{\mathcal{S}} \equiv \mathcal{S}\oplus\{\hat{\sigma}_5\}$. Obviously, $\widehat{\mathcal{S}}$ is completely integrable.

But now, we can set a particular condition defining $\widehat{\mathcal{S}}$: $\hat{\sigma}_5 \wedge d\hat{\sigma}_5 = 0$; a condition similar to those for the $\hat{\sigma}_\alpha$'s and which defines completely the functions ψ_α preserving the S_4 invariance on $\mathcal{S}$.

Moreover, we can notice that we obtain a system of geodesic coordinates $\tau_\mathfrak{h}$ on $\widehat{\mathcal{M}}$ because $\hat{\sigma}_\mathfrak{h} \wedge d\hat{\sigma}_\mathfrak{h} = 0$ for all of the indices $\mathfrak{h}$; a condition which involves that $\widehat{\mathcal{M}}$ is then necessarily conformally flat (see footnote 4).

But, we have four degrees of freedom with the functions ψ_α such that $\hat{\sigma}_5 \wedge d\hat{\sigma}_5 = 0$ and $d\hat{\sigma}_5 \neq 0$. And thus, we can set, for instance, $\psi_\alpha\,\hat{\nu}_\alpha \equiv e^{\tau_\alpha - \tau_5}$. Then, we find that $d\hat{\sigma}_5 = \hat{\sigma}_5 \wedge d\tau_5$ and

$$\hat{\sigma}_5 = d\tau_5 + e^{-\tau_5} \sum_{\alpha=1}^{4} e^{\tau_\alpha}\, d\tau_\alpha\,. \tag{9.12}$$

The advantage of such an expression for $\hat{\sigma}_5$ is that we can more easily define, as shown below, the coordinate of projection ς^0. Also, it is independent on the smooth function φ defining $\tilde{g}$ from g. Also, from the conditions $\hat{\sigma}_5(\hat{k}) = 0$ and $k^5 > 0$, we can define more precisely the fifth time stamp τ_5 and the embedding of $\mathcal{M}$ into $\widehat{\mathcal{M}}$. Indeed, setting $k^5 \equiv e^{-\tau_5}$ and $\hat{\sigma}_5(\hat{k}) = 0$, we obtain, along, for instance, a light-like curve $\gamma(\lambda) \subset \widehat{\mathcal{M}}$ with tangent light-like vector $\hat{k} \equiv (k^\mathfrak{s})$ such that $k^\alpha \equiv \frac{d\tau_\alpha}{d\lambda} \equiv \dot{\tau}_\alpha$, the relation

$$d\tau_5 = -\left(\sum_{\alpha=1}^{4} e^{\tau_\alpha}\, k^\alpha\right) d\lambda\,. \tag{9.13}$$

Then, reporting this relation within the context of the location process detailed in the previous sections, if $\gamma(\lambda = 0)$ is the *anchor* event a of a localized event $e \equiv \gamma(\lambda = 1)$, then, we deduce that

$$\tau_5 = -\int_0^1 \left(\sum_{\alpha=1}^{4} e^{\tau_\alpha}\, k^\alpha\right) d\lambda\,. \tag{9.14}$$

This relation defines the embedding of $\mathcal{M}$ into $\widehat{\mathcal{M}}$. We can note, however, that it is independent on the curve γ whatever is its type since the 1-form $\sum_{\alpha=1}^{4} e^{\tau_\alpha}\, d\tau_\alpha$ is exact. In addition, it is an illustration of the theorem 3 (p. 11).

Also, this value for τ_5 is unique if 1) only one null geodesic exists from any anchor a to its corresponding localized event e, and 2) γ is a null geodesic. The former condition is always satisfied from the causal axiomatics based on the existence of unique message functions. Then, taken γ to be the unique null geodesic from a to e, we ascribe to e the four time stamps τ_α and the fifth one τ_5 determined by the integral (9.14). Then, considering that the two conditions above are satisfied, to each localized event e corresponds a unique point $\hat{p}_e$ in a five-dimensional grid of localization such that $\hat{p}_e \equiv (\tau_{\mathfrak{k}})$. The relation (9.14) constitutes a first step towards its complete physical meaning given in the sequel. Nevertheless, we can already suggest that it is related to a notion of variation of energy since this is the only isotropic observable featuring the variation of a light ray along its pathway.

B. The projective Cartan connection and the Yano-Ishihara projecting 1-form on $\widehat{\mathcal{M}}$

In this section, we denote by $[\widehat{\mathcal{M}}]$ the projective manifold locally homeomorphic to $\mathcal{M}$, and thus, locally, we have $[\widehat{\mathcal{M}}] \simeq \mathcal{M} \simeq P\mathbb{R}^4$. To be defined, a projecting form $\hat{\pi}$ must be given on $\widehat{\mathcal{M}}$. Actually, $\hat{\pi}$ is no more no less than the 1-form $\hat{\sigma}_5$ with its condition of integrability, *i.e.*, $\hat{\sigma}_5 \wedge d\hat{\sigma}_5 = 0$, fixing the function ψ and, therefore, the metric $\hat{g}$. As a consequence, if $\hat{k} \in \widehat{\mathcal{H}}_\tau$, then, $\hat{\pi}(\hat{k}) = 0$ where $\hat{\pi} \equiv \hat{\sigma}_5$. Then, we can provide $\widehat{\mathcal{M}}$ with both a Riemannian and a projective geometry. The formalism presented below is, somehow, similar to the formalism involved into 1) the Yano-Ohgane-Ishihara unification of gravitation and electromagnetism but within the framework of projective geometry [YO52, YI67].

Also, we assume that $\widehat{\mathcal{M}}$ satisfies the conditions to be a codimension one foliation of which $\mathcal{M}$ is a particular leaf. These conditions are given p. 35 in full generality and, within the present context, they are the following:

1. $\widehat{\mathcal{M}}$ is a five-dimensional, paracompact and connected manifold of class C^r with $r \geqslant 2$,
2. the Yano-Ishihara 1-form $\hat{\pi}$ is a regular integrable 1-form of class C^{r-1} on $\widehat{\mathcal{M}}$,

3. the leaves of the foliation which are the maximal, integral, four-dimensional and connected manifolds of class C^{r-1} defined by $\hat{\pi}$ are assumed to be non-closed, and
4. there exists a complete vector field $\hat{\xi}$ of class C^{r-1} on $\widehat{\mathcal{M}}$ such that $\hat{\pi}(\hat{\xi}) = 1$.

Then, we set the following:

Definition 19. *We denote by $\widehat{\mathcal{M}}^P_{\mathcal{R}} = (\widehat{\mathcal{M}}, \widehat{\mathcal{H}}, \hat{g}, \hat{\pi})$ the five dimensional projective and Riemannian manifold $\widehat{\mathcal{M}}$ endowed with the metric $\hat{g}$ and the Yano-Ishihara projecting 1-form $\hat{\pi}$ such that*

$$\hat{g} = \tilde{g} + \hat{h}\,, \tag{9.15a}$$

$$\hat{h} = \hat{\pi} \otimes \hat{\pi}\,, \tag{9.15b}$$

$$\hat{\pi} = d\tau_5 + e^{-\tau_5} \textstyle\sum_{\alpha=1}^{4} e^{\tau_\alpha}\, d\tau_\alpha\,, \tag{9.15c}$$

and $\widehat{\mathcal{H}} = \cup_{\tau \in \widehat{\mathcal{M}}} \widehat{\mathcal{H}}_\tau \subset T\widehat{\mathcal{M}}$ where $\widehat{\mathcal{H}}_\tau \subset T_\tau \widehat{\mathcal{M}}$.

From this definition of $\hat{\pi}$, we need to know the coordinate of projection, denoted by ς^0, defined from the five parameters $\tau_{\mathfrak{h}}$. This variable can be easily defined since, from $\hat{\pi} \wedge d\hat{\pi} = 0$, there exists, necessarily, a smooth function $\varsigma^0 \in \mathcal{O}_{\widehat{\mathcal{M}}}$ such that

$$d\varsigma^0 \equiv e^{\tau_5}\, \hat{\pi}\,. \tag{9.16}$$

We find after integration that

$$\varsigma^0 = \sum_{\mathfrak{h}=1}^{5} e^{\tau_{\mathfrak{h}}}\,, \tag{9.17}$$

which can never be vanishing for finite values of the $\tau_{\mathfrak{h}}$'s. Besides, the Lie algebra of infinitesimal automorphisms preserving the 1-form $\hat{\pi}$ is the set of following commuting vector fields (see Appendix L):

$$\frac{\partial}{\partial \varsigma^{\mathfrak{k}}} = e^{-\tau_{\mathfrak{k}}}\, \frac{\partial}{\partial \tau_{\mathfrak{k}}} \tag{9.18}$$

where

$$\varsigma^{\mathfrak{k}} \equiv e^{\tau_{\mathfrak{k}}}\,, \qquad \mathfrak{k} = 1, \ldots, 5\,. \tag{9.19}$$

Furthermore, this Lie algebra is defined on the ring of univariate smooth functions depending only on ς^0.

Actually, reverting to the exact meaning of time stamps, we can say that the coordinates $\varsigma^{\mathfrak{k}}$ are just new time stamps which differ from the time stamps $\tau_{\mathfrak{k}}$ only because they must be positive whereas the $\tau_{\mathfrak{k}}$ are just assumed to be real numbers. The time stamps $\tau_{\mathfrak{k}}$ are not preferred with respect to the $\varsigma^{\mathfrak{k}}$ to parameterize the worldlines. In a way, it is just equivalent to a change of notation for the time stamps. We can rewrite the different (co)-tensors only with respect to these new variables. Therefore, we have from now the following definitions/notations:

$$\tilde{g} = -e^{-2\varphi} \sum_{\alpha<\beta=1}^{4} \tilde{\nu}_\alpha(\{\varsigma^{\mathfrak{u}}\})\, \tilde{\nu}_\beta(\{\varsigma^{\mathfrak{v}}\})\, d\varsigma^\alpha \odot d\varsigma^\beta\,, \tag{9.20a}$$

$$\hat{\pi} = \frac{1}{\varsigma^5}\, d\varsigma^0\,, \tag{9.20b}$$

$$\varsigma^0 = \sum_{\mathfrak{h}=1}^{5} \varsigma^{\mathfrak{h}}\,, \tag{9.20c}$$

$$\varsigma^{\mathfrak{u}} > 0\,, \qquad \mathfrak{u} = 1, \ldots, 5\,. \tag{9.20d}$$

where $\tilde{\nu}_\alpha(\{\varsigma^{\mathfrak{u}}\}) \equiv e^{-\tau_\alpha}\, \hat{\nu}_\alpha(\{\tau_{\mathfrak{v}}\}) > 0$.

Remark 16. *It is important to note the following. First, we denote by RLS^τ the relativistic location system made of the five emitters $\mathcal{E}$, $\overline{\mathcal{E}}$, $\widetilde{\mathcal{E}}$, $\widehat{\mathcal{E}}$ and $\mathring{\mathcal{E}}$ broadcasting their respective time stamps $\tau \equiv \tau_1$, $\bar{\tau} \equiv \tau_2$, $\tilde{\tau} \equiv \tau_3$, $\hat{\tau} \equiv \tau_4$ and $\mathring{\tau} \equiv \tau_5$ on their corresponding worldlines W, $\overline{W}$, $\widetilde{W}$, $\widehat{W}$ and $\mathring{W}$. And we denote also by RLS^ς the relativistic location system made of the same five emitters $\mathcal{E}$, $\overline{\mathcal{E}}$, $\widetilde{\mathcal{E}}$, $\widehat{\mathcal{E}}$ and $\mathring{\mathcal{E}}$ but broadcasting the respective time stamps ς^1, ς^2, ς^3, ς^4 and ς^5 on their corresponding worldlines W, $\overline{W}$, $\widetilde{W}$, $\widehat{W}$ and $\mathring{W}$ again. As we noticed, the ς's are new equivalent time stamps. We have just different parameterizations of the worldlines of the emitters. Then, contrary to the τ's, as shown in the previous sections on the protocol of localization, the new time stamps $\varsigma_{\mathfrak{k}}$ are not transformed by projective transformations defined by the RLS^τ, but, they are with respect to the new relativistic location system RLS^ς. And thus, the four new time stamps ς^α are projective coordinates with respect to RLS^ς (we consider projective transformations between the ς's on the celestial spheres rather than between their logarithms $\ln \varsigma_\alpha \equiv \tau_\alpha$).*

*The main difference between these two relativistic location systems is that we must have the constraints $\varsigma^{\mathfrak{k}} > 0$ defining RLS$^{\varsigma}$. Thenceforth, RLS$^{\varsigma}$ appears to be "*generic*" with respect to the projective structure contrary to the RLS$^{\tau}$. By "generic" we mean that the dilatation group $\mathbb{R}^*$ acts freely on the new set of time stamps $\varsigma^{\mathfrak{k}}$, i.e., it does not exit $\lambda \in \mathbb{R}^*$ such that $\lambda \neq 1$ and $\lambda\varsigma^{\mathfrak{k}} = \varsigma^{\mathfrak{k}}$. On the contrary, the isotropic group of $\tau_\alpha = 0$ is not trivial, i.e., it is equal to $\mathbb{R}^*$. Hence, we must determine the projective connection with respect to this generic RLS.*

Besides, we considered above that the five emitters of the RLS$^{\varsigma}$ broadcast the five time stamps $\varsigma^{\mathfrak{k}}$. Actually, we can consider also that they broadcast the five time stamps $\tau_{\mathfrak{k}}$ but that the projective transformations on the celestial spheres of the emitters are made with respect to their corresponding time stamps $\varsigma^{\mathfrak{k}} = e^{\tau_{\mathfrak{k}}}$.

Now, we want again to provide $\widehat{\mathcal{M}}^{P}_{\mathcal{R}}$ with both a projective and a Riemannian structure with an Euclidean Levi-Civita connection on $\widehat{\mathcal{M}}$ and a projective Cartan connection on $\mathcal{M}$. Then, in a first step, we define a coframe $\widehat{\mathcal{B}}^*$ such that

$$\widehat{\mathcal{B}}^* \equiv \{\hat{\pi}^1 \equiv (1/\varsigma^5)\, d\varsigma^1, ..., \hat{\pi}^4 \equiv (1/\varsigma^5)\, d\varsigma^4, \hat{\pi}^5 \equiv \hat{\pi} = (1/\varsigma^5)\, d\varsigma^0\} \tag{9.21}$$

and its dual frame

$$\widehat{\mathcal{B}} \equiv \{\hat{\xi}_1, ..., \hat{\xi}_5\} \tag{9.22}$$

such that

$$\hat{\pi}^{\mathfrak{k}}(\hat{\xi}_{\mathfrak{r}}) = \delta^{\mathfrak{k}}_{\mathfrak{r}}\,. \tag{9.23}$$

We obtain ($\alpha = 1, \dots, 4$):

$$\hat{\xi}_5 = \varsigma^5\, \partial_5\,, \tag{9.24a}$$

$$\hat{\xi}_\alpha = \varsigma^5\, (\partial_\alpha - \partial_5)\,, \tag{9.24b}$$

where $\partial_{\mathfrak{h}} \equiv \partial\ /\partial\varsigma^{\mathfrak{h}}$, and it follows that

$$\hat{g}(\hat{\xi}_5, \hat{\xi}_5) \equiv \hat{g}_{55} = 1\,, \tag{9.25a}$$

$$\hat{g}(\hat{\xi}_\alpha, \hat{\xi}_5) \equiv \hat{g}_{\alpha 5} = \hat{g}_{5\alpha} = 0\,, \tag{9.25b}$$

$$\hat{g}(\hat{\xi}_\alpha, \hat{\xi}_\beta) \equiv \hat{g}_{\alpha\beta} = \frac{1}{2}\, e^{-2\varphi}\, \tilde{\nu}_\alpha\, \tilde{\nu}_\beta\, (\varsigma^5)^2\, (\delta_{\alpha\beta} - 1)\,. \tag{9.25c}$$

Hence, $\widehat{\mathcal{B}}$ is of type $\{s\ell\ell\ell\ell\}$ and it is a solvable Lie algebra with a four-dimensional abelian nilradical $\{\hat{\xi}_1 + \hat{\xi}_5, \ldots, \hat{\xi}_4 + \hat{\xi}_5\}$. The nonvanishing commutators are the following:

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] = \hat{\xi}_\alpha - \hat{\xi}_\beta\,, \qquad [\hat{\xi}_5, \hat{\xi}_\alpha] = \hat{\xi}_\alpha + \hat{\xi}_5\,. \tag{9.26}$$

Hence, $\widehat{\mathcal{B}}$ is a *nonholonomic* frame. Additionally, the 1-forms $\hat{\pi}^{\mathfrak{k}}$ satisfy the following set of Frobenius relations ($q_\alpha = 1$ for all $\alpha = 1, \ldots, 4$):

$$d\hat{\pi}^5 = \left(\sum_{\alpha=1}^{4} q_\alpha\, \hat{\pi}^\alpha\right) \wedge \hat{\pi}^5\,, \tag{9.27a}$$

$$d\hat{\pi}^\alpha = \hat{\pi}^\alpha \wedge \left(\hat{\pi}^5 - \sum_{\beta=1}^{4} q_\beta\, \hat{\pi}^\beta\right). \tag{9.27b}$$

Furthermore, in the non-commutative frame $\widehat{\mathcal{B}}$ and coframe $\widehat{\mathcal{B}}^*$, the coefficients of the Levi-Civita connection $\widehat{\Gamma}$ are obtained from the formula:

$$\widehat{\Gamma}^{\mathfrak{u}}_{\mathfrak{s},\mathfrak{h}} = \frac{1}{2}\sum_{\mathfrak{v}=1}^{5} \hat{g}^{\mathfrak{u}\mathfrak{v}}\Big\{\hat{g}(\hat{\xi}_{\mathfrak{v}}, [\hat{\xi}_{\mathfrak{h}}, \hat{\xi}_{\mathfrak{s}}]) + \hat{g}(\hat{\xi}_{\mathfrak{h}}, [\hat{\xi}_{\mathfrak{v}}, \hat{\xi}_{\mathfrak{s}}]) + \hat{g}(\hat{\xi}_{\mathfrak{s}}, [\hat{\xi}_{\mathfrak{v}}, \hat{\xi}_{\mathfrak{h}}]) + \hat{\xi}_{\mathfrak{h}}(\hat{g}_{\mathfrak{v}\mathfrak{s}}) + \hat{\xi}_{\mathfrak{s}}(\hat{g}_{\mathfrak{v}\mathfrak{h}}) - \hat{\xi}_{\mathfrak{v}}(\hat{g}_{\mathfrak{s}\mathfrak{h}})\Big\}, \tag{9.28}$$

where we identified the Lie derivative $\mathcal{L}_{\hat{\xi}_{\mathfrak{h}}}$ with $\hat{\xi}_{\mathfrak{h}}$. Also, the relations

$$\widehat{\Gamma} \equiv \sum_{\mathfrak{h},\mathfrak{s},\mathfrak{u}=1}^{5} \widehat{\Gamma}^{\mathfrak{u}}_{\mathfrak{h},\mathfrak{s}}\; \hat{\pi}^{\mathfrak{s}} \otimes \hat{\xi}_{\mathfrak{u}} \otimes \hat{\pi}^{\mathfrak{h}}\,, \qquad \widehat{\nabla}_{\hat{\xi}_{\mathfrak{s}}} \hat{\xi}_{\mathfrak{h}} \equiv \sum_{\mathfrak{u}=1}^{5} \widehat{\Gamma}^{\mathfrak{u}}_{\mathfrak{h},\mathfrak{s}}\, \hat{\xi}_{\mathfrak{u}}\,, \qquad \widehat{\nabla}_{\hat{\xi}_{\mathfrak{s}}} \hat{\pi}^{\mathfrak{h}} \equiv -\sum_{\mathfrak{h}=1}^{5} \widehat{\Gamma}^{\mathfrak{h}}_{\mathfrak{k},\mathfrak{s}}\, \hat{\pi}^{\mathfrak{k}} \tag{9.29}$$

hold where $\widehat{\nabla}$ corresponds to the covariant derivative defined by $\widehat{\Gamma}$. Obviously, from the definition of the Levi-Civita connection, the compatibility condition for $\hat{g}$ is necessarily satisfied, *i.e.*, we have:

$$\widehat{\nabla}\hat{g} = 0\,. \tag{9.30}$$

We denote also by $\widehat{\Gamma}^{\mathfrak{h}}_{\mathfrak{k}}$ the symbols such that

$$\widehat{\Gamma}^{\mathfrak{h}}_{\mathfrak{k}} \equiv \sum_{\mathfrak{r}=1}^{5} \widehat{\Gamma}^{\mathfrak{h}}_{\mathfrak{k},\mathfrak{r}}\, \hat{\pi}^{\mathfrak{r}}\,, \tag{9.31}$$

and thus, from the compatibility condition for $\hat{g}$, we obtain that

$$d\hat{g}_{\mathfrak{k}\mathfrak{h}} = \sum_{\mathfrak{r}=1}^{5} \left(\hat{g}_{\mathfrak{k}\mathfrak{r}}\, \widehat{\Gamma}^{\mathfrak{r}}_{\mathfrak{h}} + \hat{g}_{\mathfrak{h}\mathfrak{r}}\, \widehat{\Gamma}^{\mathfrak{r}}_{\mathfrak{k}}\right). \tag{9.32}$$

In particular, if $\mathfrak{k} = \alpha$ and $\mathfrak{h} = 5$, we deduce a first important relation ($\alpha = 1, \ldots, 4$):

$$\sum_{\beta=1}^{4} \hat{g}_{\alpha\beta}\, \widehat{\Gamma}^{\beta}_{5} + \widehat{\Gamma}^{5}_{\alpha} = 0\,, \tag{9.33}$$

which is supplemented by a second relation obtained with $\mathfrak{k} = \mathfrak{h} = 5$:

$$\widehat{\Gamma}^{5}_{5} = 0\,. \tag{9.34}$$

Besides, from (9.28), we can see easily that the relations

$$\widehat{\Gamma}^{\mathfrak{u}}_{\mathfrak{h},\mathfrak{r}} - \widehat{\Gamma}^{\mathfrak{u}}_{\mathfrak{r},\mathfrak{h}} = C^{\mathfrak{u}}_{\mathfrak{r},\mathfrak{h}} \tag{9.35}$$

hold, where the $C^{\mathfrak{u}}_{\mathfrak{r},\mathfrak{h}}$'s are the structure constants such that

$$[\hat{\xi}_{\mathfrak{r}},\, \hat{\xi}_{\mathfrak{h}}] = \sum_{\mathfrak{u}=1}^{5} C^{\mathfrak{u}}_{\mathfrak{r},\mathfrak{h}}\, \hat{\xi}_{\mathfrak{u}}\,. \tag{9.36}$$

Then, from (9.35) with $\mathfrak{u} = \mathfrak{h} = 5$ and $\mathfrak{r} = \alpha$, we obtain the fundamental result:

$$\widehat{\Gamma}^{5}_{\alpha,5} = 1\,. \tag{9.37}$$

As an important consequence, this result forbids to consider $\hat{\xi}_5$ as the transversal vector $\hat{\xi}$ such that $i_{\hat{\xi}}\hat{\pi} \equiv \hat{\pi}(\hat{\xi}) = 1$. Indeed, if we consider $\hat{\xi} \equiv \hat{\xi}^5$, then, from the second relation (9.34), *i.e.*, $\widehat{\Gamma}^5_5 = 0$, we deduce that the projective Cartan connection ω is equal to $\widehat{\Gamma}$, *i.e.*, $\omega \equiv \widehat{\Gamma}$. And then, the two constraints $i_{\hat{\xi}}\widehat{\Gamma}^{\mathfrak{k}}_{\mathfrak{h}} = 0$ and $Tr(\widehat{\Gamma}) = 0$ must be imposed. But, the former constraint cannot be satisfied since we have, in particular, $i_{\hat{\xi}}\widehat{\Gamma}^5_\alpha = i_{\hat{\xi}^5}\widehat{\Gamma}^5_\alpha = \widehat{\Gamma}^5_{\alpha,5} \neq 0$. And thus, the projective Cartan connection ω must differ from $\widehat{\Gamma}$.

Hence, in full generality, we define $\hat{\xi}$ such that

$$\hat{\xi} = \sum_{\mathfrak{h}=1}^{5} v^{\mathfrak{h}}\, \hat{\xi}_{\mathfrak{h}}\,, \qquad v^5 = 1\,, \tag{9.38}$$

where, at least, one coordinate v^α exists such that $v^\alpha \neq 0$. Then, in a second step, we define ω from the Levi-Civita connection $\mathring{\Gamma}$ obtained in the new frame $\mathring{\mathcal{B}}$ with its dual coframe $\mathring{\mathcal{B}}^*$ such that

$$\mathring{\mathcal{B}} = \{\mathring{\xi}_1, \ldots, \mathring{\xi}_4, \mathring{\xi}_5 \equiv \hat{\xi}\}\,, \tag{9.39}$$

$$\mathring{\mathcal{B}}^* = \{\mathring{\pi}^1, \ldots, \mathring{\pi}^4, \mathring{\pi}^5 \equiv \hat{\pi}\} \tag{9.40}$$

where

$$\mathring{\xi}_\alpha = \hat{\xi}_\alpha\,, \qquad \text{for all} \quad \alpha = 1,\ldots,4, \tag{9.41a}$$

$$\mathring{\xi}_5 \equiv \hat{\xi} = \hat{\xi}_5 + \sum_{\alpha=1}^{4} v^\alpha\,\hat{\xi}_\alpha\,, \tag{9.41b}$$

$$[\mathring{\xi}_\alpha, \mathring{\xi}_\beta] = \mathring{\xi}_\alpha - \mathring{\xi}_\beta\,, \tag{9.41c}$$

$$[\mathring{\xi}_5, \mathring{\xi}_\alpha] = \mathring{\xi}_5 - \Theta\,\mathring{\xi}_\alpha - \sum_{\mu=1}^{4} \mathring{\xi}_\alpha(v^\mu)\,\mathring{\xi}_\mu\,, \quad (q_\mu = 1 \text{ for } \mu = 1,\ldots,4), \tag{9.41d}$$

$$\Theta \equiv \sum_{\rho=1}^{4} q_\rho\,v^\rho - 1\,, \tag{9.41e}$$

and

$$\mathring{\pi}^\alpha = \hat{\pi}^\alpha - v^\alpha\,\hat{\pi}^5\,, \qquad \text{for all} \quad \alpha = 1,\ldots,4, \tag{9.42a}$$

$$\mathring{\pi}^5 = \hat{\pi}^5 \equiv \hat{\pi}\,, \tag{9.42b}$$

$$d\mathring{\pi}^5 = \left(\sum_{\mu=1}^{4} q_\mu\,\mathring{\pi}^\mu\right) \wedge \mathring{\pi}^5\,, \tag{9.42c}$$

$$d\mathring{\pi}^\alpha = \mathring{\pi}^5 \wedge \left(dv^\alpha + \Theta\,\mathring{\pi}^\alpha\right) + \left(\sum_{\mu=1}^{4} q_\mu\,\mathring{\pi}^\mu\right) \wedge \mathring{\pi}^\alpha\,. \tag{9.42d}$$

Also, we define the structure constants $\mathring{C}^{\mathfrak{k}}_{\mathfrak{u},\mathfrak{v}}$ such that

$$[\mathring{\xi}_{\mathfrak{u}}, \mathring{\xi}_{\mathfrak{v}}] = \sum_{\mathfrak{k}=1}^{5} \mathring{C}^{\mathfrak{k}}_{\mathfrak{u},\mathfrak{v}}\,\mathring{\xi}_{\mathfrak{k}}\,. \tag{9.43}$$

And we have, in particular, the following relations between the coefficients $\mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{v}}$:

$$\mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{v},\mathfrak{u}} - \mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{u},\mathfrak{v}} = \mathring{C}^{\mathfrak{k}}_{\mathfrak{u},\mathfrak{v}}\,. \tag{9.44}$$

Now, we present below a first set of useful and important formulas, preliminary to the definition of the projective Cartan connection ω given in the sequel.

In particular, we have $\mathring{\pi}^{\mathfrak{k}}(\mathring{\xi}_{\mathfrak{h}}) = \delta^{\mathfrak{k}}_{\mathfrak{h}}$ from $\hat{\pi}^{\mathfrak{k}}(\hat{\xi}_{\mathfrak{h}}) = \delta^{\mathfrak{k}}_{\mathfrak{h}}$ for all $\mathfrak{k},\mathfrak{h} = 1,\ldots,5$. Also, we obtain the following definitions and relations:

$$\mathring{\Gamma} \equiv \sum_{\mathfrak{k},\mathfrak{h},\mathfrak{u}=1}^{5} \mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{k},\mathfrak{h}}\, \mathring{\pi}^{\mathfrak{h}} \otimes \mathring{\xi}_{\mathfrak{u}} \otimes \mathring{\pi}^{\mathfrak{k}}\,, \tag{9.45a}$$

$$\mathring{\Gamma}^{\mathfrak{h}}_{\mathfrak{k}} \equiv \sum_{\mathfrak{u}=1}^{5} \mathring{\Gamma}^{\mathfrak{h}}_{\mathfrak{k},\mathfrak{u}}\, \mathring{\pi}^{\mathfrak{u}}\,, \tag{9.45b}$$

$$\widehat{\nabla} \mathring{\xi}_{\mathfrak{k}} = \sum_{\mathfrak{u}=1}^{5} \mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{k}} \otimes \mathring{\xi}_{\mathfrak{u}}\,, \tag{9.45c}$$

$$\widehat{\nabla} \mathring{\pi}^{\mathfrak{k}} = -\sum_{\mathfrak{u}=1}^{5} \mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{u}} \otimes \mathring{\pi}^{\mathfrak{u}}\,. \tag{9.45d}$$

Also, we denote by K the change-of-basis matrix from $\widehat{\mathcal{B}}$ to $\mathring{\mathcal{B}}$ such that $\mathring{\xi}_{\mathfrak{k}} = \sum_{\mathfrak{u}=1}^{5} K^{\mathfrak{u}}_{\mathfrak{k}}\, \hat{\xi}_{\mathfrak{u}}$. Then, we obtain $K \equiv \mathbb{1} + L$ where $L^{\mathfrak{u}}_{\mathfrak{k}} \equiv \delta^5_{\mathfrak{k}} \big(\sum_{\alpha=1}^{4} \delta^{\mathfrak{u}}_{\alpha}\, v^{\alpha} \big)$ and

$$\begin{pmatrix} \mathring{\xi}_1 \\ \mathring{\xi}_2 \\ \mathring{\xi}_3 \\ \mathring{\xi}_4 \\ \mathring{\xi}_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ v^1 & v^2 & v^3 & v^4 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \\ \hat{\xi}_3 \\ \hat{\xi}_4 \\ \hat{\xi}_5 \end{pmatrix}. \tag{9.46}$$

Consequently, we deduce the following relation between $\widehat{\Gamma}$ and $\mathring{\Gamma}$:

$$\mathring{\Gamma} = \widehat{\Gamma} + dL + [\widehat{\Gamma}, L] - L\, \widehat{\Gamma}\, L\,, \tag{9.47}$$

or, equivalently, the relations:

$$\mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{h}} = \widehat{\Gamma}^{\mathfrak{k}}_{\mathfrak{h}} + \delta^5_{\mathfrak{h}} \sum_{\alpha=1}^{4} \left[\delta^{\mathfrak{k}}_{\alpha} \left(dv^{\alpha} - \left\{ \sum_{\beta=1}^{4} \widehat{\Gamma}^5_{\beta}\, v^{\beta} \right\} v^{\alpha} \right) + \widehat{\Gamma}^{\mathfrak{k}}_{\alpha}\, v^{\alpha} \right] - \widehat{\Gamma}^5_{\mathfrak{h}} \left(\sum_{\alpha=1}^{4} \delta^{\mathfrak{k}}_{\alpha}\, v^{\alpha} \right). \tag{9.48}$$

In particular, we obtain the equality between the traces:

$$Tr(\mathring{\Gamma}) = Tr(\widehat{\Gamma})\,. \tag{9.49}$$

Furthermore, denoting by $\mathring{g}_{\mathfrak{k}\mathfrak{h}}$ the coefficients deduced from the metric $\hat{g}$ such that $\mathring{g}_{\mathfrak{k}\mathfrak{h}} = \mathring{g}_{\mathfrak{h}\mathfrak{k}} =$

$\hat{g}(\mathring{\xi}_{\mathfrak{k}}, \mathring{\xi}_{\mathfrak{h}})$, we obtain the following defining relations ($q_\alpha = 1$ for all $\alpha = 1, \ldots, 4$):

$$\mathring{g}_{\alpha\beta} = \hat{g}_{\alpha\beta} = \tilde{g}_{\alpha\beta} + q_\alpha\, q_\beta\,, \tag{9.50a}$$

$$\mathring{g}_{55} = 1 + \sum_{\alpha,\beta=1}^{4} \mathring{g}_{\alpha\beta}\, v^\alpha\, v^\beta\,, \tag{9.50b}$$

$$\mathring{g}_{\alpha 5} = \sum_{\beta=1}^{4} \mathring{g}_{\alpha\beta}\, v^\beta\,. \tag{9.50c}$$

And, moreover, from the relations $\delta^\alpha_5 = \sum_{\mathfrak{h}=1}^5 \mathring{g}^{\alpha\mathfrak{h}}\, \mathring{g}_{\mathfrak{h}5}$ and $\delta^5_\alpha = \sum_{\mathfrak{h}=1}^5 \mathring{g}^{5\mathfrak{h}}\, \mathring{g}_{\mathfrak{h}\alpha}$ we deduce that the coefficients $\mathring{g}^{\mathfrak{u}\mathfrak{h}}$ ($= \mathring{g}^{\mathfrak{h}\mathfrak{u}}$) of the inverse metric of $\hat{g}$ in the frame $\mathring{\mathcal{B}}$ are such that

$$\mathring{g}^{\alpha 5} = -v^\alpha\,, \tag{9.51a}$$

$$\mathring{g}^{55} = 1\,. \tag{9.51b}$$

Also, we define the coefficients $\breve{g}^{\alpha\beta}$ such that

$$\sum_{\mu=1}^{4} \mathring{g}_{\beta\mu}\, \breve{g}^{\mu\alpha} = \delta^\alpha_\beta\,. \tag{9.52}$$

Then, from the relation $\sum_{\mathfrak{h}=1}^5 \mathring{g}_{\beta\mathfrak{h}}\, \mathring{g}^{\mathfrak{h}\alpha} = \delta^\alpha_\beta$, we deduce the following additional relations:

$$\breve{g}^{\alpha\beta} \equiv \mathring{g}^{\alpha\beta} - v^\alpha\, v^\beta\,, \tag{9.53a}$$

$$\sum_{\mu=1}^{4} \mathring{g}_{\beta\mu}\, \mathring{g}^{\alpha\mu} = \delta^\alpha_\beta + v^\alpha \left(\sum_{\mu=1}^{4} \mathring{g}_{\beta\mu}\, v^\mu \right). \tag{9.53b}$$

Lastly, to finish this first set of formulas, the compatibility condition $\widehat{\nabla}\hat{g} = 0$ is equivalent to the formulas:

$$d\mathring{g}_{\mathfrak{h}\mathfrak{k}} = \sum_{\mathfrak{u}=1}^{5} \left(\mathring{g}_{\mathfrak{h}\mathfrak{u}}\, \mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{k}} + \mathring{g}_{\mathfrak{k}\mathfrak{u}}\, \mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{h}} \right), \tag{9.54a}$$

$$d\mathring{g}^{\mathfrak{h}\mathfrak{k}} = -\sum_{\mathfrak{u}=1}^{5} \left(\mathring{g}^{\mathfrak{h}\mathfrak{u}}\, \mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{u}} + \mathring{g}^{\mathfrak{k}\mathfrak{u}}\, \mathring{\Gamma}^{\mathfrak{h}}_{\mathfrak{u}} \right). \tag{9.54b}$$

Then, if we take successively $\mathfrak{h} = \mathfrak{k} = 5$ in (9.54b) and $\mathfrak{h} = \alpha$ with $\mathfrak{k} = 5$ in (9.54a), and because $\mathring{g}^{55} = 1$ and $\mathring{g}_{\alpha 5} = \sum_{\beta=1}^4 \mathring{g}_{\alpha\beta}\, v^\beta$, we obtain the two following fundamental formulas:

$$\mathring{\Gamma}^5_5 = \sum_{\alpha=1}^{4} \mathring{\Gamma}^5_\alpha\, v^\alpha\,, \tag{9.55a}$$

$$\mathring{\Gamma}^5_\alpha = \sum_{\beta=1}^{4} \mathring{g}_{\alpha\beta} \left(dv^\beta - \mathring{\Gamma}^\beta_5 + \sum_{\mu=1}^{4} \mathring{\Gamma}^\beta_\mu\, v^\mu \right). \tag{9.55b}$$

The last relation (9.55b) can also be obtained starting with $\mathfrak{h} = \alpha$ and $\mathfrak{k} = 5$ in (9.54b) considering that $\mathring{g}^{\alpha 5} = -v^{\alpha}$.

1. The projective Cartan connection ω

Actually, we define only a (pre-)projective Cartan connection $\mathring{\omega}$ from $\mathring{\Gamma}$ (see Def. 13, p. 61) and not a $\mathring{\mathcal{B}}$-complete projective Cartan connection. The projective connection $\mathring{\omega}$ is defined by the following formula in the frame $\mathring{\mathcal{B}}$:

$$\mathring{\omega} \equiv \mathring{\Gamma} - \mathring{\Gamma}^5_5\,\mathbb{1}\,, \tag{9.56}$$

where

$$\hat{\pi}(\,\mathring{\Gamma}.\,\hat{\xi}) \equiv \mathring{\pi}^5(\,\mathring{\Gamma}.\,\mathring{\xi}_5) \equiv \mathring{\Gamma}^5_5\,. \tag{9.57}$$

Now, the projective connection $\mathring{\omega}$ is a (pre-)projective Cartan connection if the following list of conditions is satisfied:

1. $\hat{\pi}(\mathring{\omega}.\,\hat{\xi}) = \mathring{\omega}^5_5 = 0.$
2. $Tr(\mathring{\omega}) = 0.$
3. There exists a vector field $\hat{\xi}$ defining $\widehat{\mathcal{M}}$ as a codimension one foliation with $\mathcal{M}$ as a leaf, and such that
   (a) $i_{\hat{\xi}}\,\mathring{\omega} = 0,$
   (b) $i_{\hat{\xi}}\hat{\pi} = 1.$

a. ***The condition 1.*** This condition is automatically satisfied from the definition (9.56) of $\mathring{\omega}$.

b. ***The condition 2.*** Setting $Tr(\mathring{\omega}) = 0$ is equivalent to set $Tr(\mathring{\Gamma}) = 5\,\mathring{\Gamma}^5_5$. We can obtain $Tr(\mathring{\Gamma})$ from $Tr(\widehat{\Gamma})$ since $Tr(\mathring{\Gamma}) = Tr(\widehat{\Gamma})$. More precisely, we can deduce the trace $Tr(\widehat{\Gamma})$ from the compatibility condition $\widehat{\nabla}|\hat{g}| = d|\hat{g}| = 0$ where

$$|\hat{g}| \equiv |\det \hat{g}_{\mathcal{B}}|$$

is the absolute value of the determinant $\det \hat{g}_{\mathcal{B}}$ of $\hat{g}$ in the coframe $\widehat{\mathcal{B}}^*$ such that (see Appendix M, formula (M22), p. 243)

$$\det \hat{g}_{\mathcal{B}} = -\frac{3}{16}\, e^{-2\left(4\varphi + \sum_{\alpha=1}^{4}(\tau_\alpha - \tau_5)\right)} \left(\prod_{\beta=1}^{4} \hat{\nu}_\beta\right)^2 . \tag{9.58}$$

This scalar field $|\hat{g}|$ is a scalar density of weight 2, and thus, in full generality, we have

$$d|\hat{g}| = 2\,Tr(\widehat{\Gamma})\,|\hat{g}| . \tag{9.59}$$

Therefore, $Tr(\widehat{\Gamma})$ is a scalar density of weight 1 such that

$$Tr(\widehat{\Gamma}) = \frac{1}{2}\, d\left(\ln|\hat{g}|\right) . \tag{9.60}$$

Thenceforth, we have the following defining relation for $\mathring{\Gamma}^5_5$:

$$\mathring{\Gamma}^5_5 = \frac{1}{10}\, d\left(\ln|\hat{g}|\right) . \tag{9.61}$$

c. ***The condition 3.*** The relation $i_{\hat{\xi}}\hat{\pi} = 1$ is satisfied by definition of $\hat{\pi}$ and $\hat{\xi}$. Thus, we must only impose the relation $i_{\hat{\xi}}\,\mathring{\omega} = 0$. It is equivalent to the following set of expressions ($\mathfrak{k}, \mathfrak{h} = 1, \ldots, 5$):

$$i_{\hat{\xi}}\,\mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{h}} = 0 \qquad \text{whenever } \mathfrak{k} \neq \mathfrak{h} , \tag{9.62a}$$

$$i_{\hat{\xi}}\left(\mathring{\Gamma}^{\mathfrak{h}}_{\mathfrak{h}} - \mathring{\Gamma}^5_5\right) = 0 . \tag{9.62b}$$

In particular, we have $i_{\hat{\xi}}\mathring{\Gamma}^5_\alpha \equiv \mathring{\Gamma}^5_{\alpha,5} = 0$. And from the relation (9.55a), *i.e.*, $\mathring{\Gamma}^5_5 = \sum_{\alpha=1}^4 \mathring{\Gamma}^5_\alpha\, v^\alpha$, we deduce that $i_{\hat{\xi}}\mathring{\Gamma}^5_5 = 0$. Therefore, the condition 3 reduces to the general relations:

$$i_{\hat{\xi}}\,\mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{h}} \equiv \mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{h},5} = 0 \tag{9.63}$$

for all $\mathfrak{k}$ and $\mathfrak{h}$ from 1 to 5. As a particular result from this constraint on $\mathring{\Gamma}$ defining $\mathring{\omega}$, we find that

$$i_{\hat{\xi}}\, dv^{\alpha} \equiv \mathring{\xi}_5(v^{\alpha}) = 0\,. \tag{9.64}$$

Indeed, from (9.63), we obtain necessarily $i_{\hat{\xi}}\, d\mathring{g}_{\mathfrak{h}\mathfrak{k}} = 0$ and $i_{\hat{\xi}}\, d\mathring{g}^{\mathfrak{h}\mathfrak{k}} = 0$ from the relations (9.55). Then, we just consider $i_{\hat{\xi}}\, d\mathring{g}^{\alpha 5} = 0$ with $\mathring{g}^{\alpha 5} = -v^{\alpha}$.

In addition, we have the relations $\mathring{\Gamma}^5_{5,\alpha} = \mathring{\Gamma}^5_{\alpha,5} + \mathring{C}^5_{\alpha,5} = 0 + (-1) = -1$ and $\mathring{\Gamma}^{\alpha}_{5,\beta} = \mathring{\Gamma}^{\alpha}_{\beta,5} + \mathring{C}^{\alpha}_{\beta,5} = 0 + \Theta\, \delta^{\alpha}_{\beta} + \mathring{\xi}_{\beta}(v^{\alpha})$. Therefore, we deduce also the following important formulas used in the sequel:

$$\mathring{\Gamma}^{\alpha}_5 = \Theta\, \mathring{\pi}^{\alpha} + \sum_{\mu=1}^{4} \mathring{\xi}_{\mu}(v^{\alpha})\, \mathring{\pi}^{\mu}\,, \tag{9.65a}$$

$$\mathring{\Gamma}^5_5 = -\sum_{\alpha=1}^{4} q_{\alpha}\, \mathring{\pi}^{\alpha}\,. \tag{9.65b}$$

In particular, the relations (9.65b) and (9.61) define completely the function φ up to a constant factor since we deduce that $d(\ln|\hat{g}|) = -10 \sum_{\alpha=1}^{4} q_{\alpha}\, \mathring{\pi}^{\alpha}$, *i.e.*, the relations

$$\mathring{\xi}_5(|\hat{g}|) = 0\,, \tag{9.66a}$$

$$\mathring{\xi}_{\alpha}(\ln|\hat{g}|) = -10 \qquad \text{for all } \alpha = 1, \ldots, 4 \tag{9.66b}$$

must hold.

As a first result, we found in this first section two important sets of relations, *viz.*, the algebraic expressions (9.55) between the coefficients $\mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{h}}$ and, in addition, the defining relation (9.61) for $\mathring{\Gamma}^5_5$. Hence, the coefficients $\mathring{\Gamma}^{\mathfrak{k}}_{\mathfrak{h}}$ are not yet completely defined from $\hat{g}$. As we shall see in the sequel, $\mathring{\Gamma}$ is, actually, completely defined from given physical tensor fields. The first one is the Faraday tensor deduced from the equations of the projective geodesics defined by $\mathring{\omega}$ (see Appendix N for a general presentation of these projective geodesic equations).

2. The projective geodesic equations

Let $\mathring{u}$ be a 5-vector such that

$$\mathring{u} = \sum_{\alpha=1}^{4} \mathring{u}^{\alpha}\, \mathring{\xi}_{\alpha} - k\, \mathring{\xi}_5\,, \tag{9.67}$$

where k is a nonvanishing constant. Then, $\mathring{u}$ is the tangent vector of a projective geodesic if and only if (see Appendix N for a general presentation) the covariant derivative of $\mathring{u}$ is such that

$$\widehat{\nabla}_{\mathring{u}}\mathring{u} \equiv \theta\,\mathring{\xi}_5 + \lambda\,\mathring{u}\,, \tag{9.68}$$

where θ and λ are any functions defined on $T\widehat{\mathcal{M}}$. The geodesic equation (9.68) can be written equivalently in the system of coordinates u^α as the following system of ODEs:

$$i_{\mathring{u}}d\mathring{u}^\alpha + \sum_{\beta,\mu=1}^{4} \mathring{\Gamma}^\alpha_{\beta,\mu}\,\mathring{u}^\beta\,\mathring{u}^\mu - k\sum_{\sigma=1}^{4}\mathring{\Gamma}^\alpha_{5,\sigma}\,\mathring{u}^\sigma = \lambda\,\mathring{u}^\alpha\,. \tag{9.69}$$

From (9.65a), we can take λ such that

$$\lambda \equiv -k\,\Theta. \tag{9.70}$$

And then, the system of geodesic equations becomes:

$$i_{\mathring{u}}d\mathring{u}^\alpha + \sum_{\beta,\mu=1}^{4} \mathring{\Gamma}^\alpha_{\beta,\mu}\,\mathring{u}^\beta\,\mathring{u}^\mu = k\sum_{\mu=1}^{4}\mathring{\xi}_\mu(v^\alpha)\,\mathring{u}^\mu\,. \tag{9.71}$$

Then, let $\mathring{\mathcal{A}}$ and $\mathring{\mathfrak{F}}$ be, respectively, a 1-form and a 2-form on $\widehat{\mathcal{M}}$ such that (considering that $\hat{\xi}_5(v^\alpha)=0$)

$$\mathring{\mathcal{A}}^\alpha_\beta \equiv \mathring{\xi}_\beta(v^\alpha)\,, \qquad \mathring{\mathcal{A}}_{\alpha,\beta} \equiv \sum_{\mu=1}^{4}\mathring{g}_{\alpha\mu}\,\mathring{\mathcal{A}}^\mu_\beta\,, \qquad \mathring{\mathcal{A}}^\alpha_\beta \equiv \sum_{\mu=1}^{4}\breve{g}^{\alpha\mu}\,\mathring{\mathcal{A}}_{\mu,\beta}\,, \tag{9.72a}$$

$$\mathring{\mathfrak{F}} \equiv \sum_{\alpha,\beta=1}^{4}\mathring{\mathcal{A}}_{\alpha,\beta}\,\mathring{\pi}^\alpha\wedge\mathring{\pi}^\beta\,. \tag{9.72b}$$

Then, $\mathring{\mathfrak{F}}$ can be also written as

$$\mathring{\mathfrak{F}} = \sum_{\alpha<\beta=1}^{4}\mathring{\mathfrak{F}}_{\alpha,\beta}\,\mathring{\pi}^\alpha\wedge\mathring{\pi}^\beta\,, \tag{9.73}$$

where

$$\mathring{\mathfrak{F}}_{\alpha,\beta} \equiv \mathring{\mathcal{A}}_{\alpha,\beta} - \mathring{\mathcal{A}}_{\beta,\alpha}. \tag{9.74}$$

Now, we can notice that the four variables v^α have not yet been really defined. There are not really constrained by the five relations $\mathring{\pi}^{\mathfrak{h}}(\mathring{\xi}_5)=0$ which are trivially satisfied by definition of

$\mathring{\xi}_5$ independently on the values taken by the v^α's. Only, the conditions $\mathring{\xi}_5(v^\alpha) = 0$ for all of the α came out from the defining constraints (the three previous conditions 1, 2 and 3) of the (pre-)projective Cartan connection $\mathring{\omega}$. The relations $\mathring{\pi}^{\mathfrak{h}}(\mathring{\xi}_5) = 0$ constraint only the variations of the v^α's with respect to at most one variable among the five coordinates $\varsigma^{\mathfrak{k}}$. Consequently, we can impose some suitable, additional constraints on the variables v^α's. In fact, we impose the following natural conditions between the four other derivatives: $\sum_{\mu=1}^4 \left(\mathring{g}_{\alpha\mu}\, \mathring{\xi}_\beta(v^\mu) + \mathring{g}_{\beta\mu}\, \mathring{\xi}_\alpha(v^\mu)\right) = 0$, *i.e.*, we impose that

$$\mathring{\mathcal{A}}_{\alpha,\beta} \equiv -\mathring{\mathcal{A}}_{\beta,\alpha}\,. \tag{9.75}$$

And as a consequence, $\mathring{\mathfrak{F}}$ can be considered as the Faraday tensor only if $d\mathring{\mathfrak{F}} = 0$, *i.e.*, only if $\mathring{\mathfrak{F}}$ is a closed 2-form. More precisely, differentiating $\mathring{\mathfrak{F}}$, we obtain the following expressions:

$$\begin{aligned} d\mathring{\mathfrak{F}} &= \sum_{\alpha,\beta=1}^4 \left(d\mathring{\mathcal{A}}_{\alpha,\beta} \wedge \mathring{\pi}^\alpha \wedge \mathring{\pi}^\beta + \mathring{\mathcal{A}}_{\alpha,\beta}\left[d\mathring{\pi}^\alpha \wedge \mathring{\pi}^\beta - \mathring{\pi}^\alpha \wedge d\mathring{\pi}^\beta\right]\right), \\ &= \sum_{\alpha,\beta=1}^4 \left(d\mathring{\mathcal{A}}_{\alpha,\beta} \wedge \mathring{\pi}^\alpha \wedge \mathring{\pi}^\beta + 2\,\mathring{\mathcal{A}}_{\alpha,\beta}\, d\mathring{\pi}^\alpha \wedge \mathring{\pi}^\beta\right). \end{aligned}$$

And from 1) the relations (9.42d), 2) the definition of the differentials $dv^\alpha \equiv \sum_{\beta=1}^4 \mathring{\xi}_\beta(v^\alpha)\,\mathring{\pi}^\beta$ because $\mathring{\xi}_5(v^\alpha) = 0$ and 3) the differentials $d\mathring{\mathcal{A}}_{\alpha,\beta} \equiv \mathring{\xi}_5(\mathring{\mathcal{A}}_{\alpha,\beta})\,\mathring{\pi}^5 + \sum_{\mu=1}^4 \mathring{\xi}_\mu(\mathring{\mathcal{A}}_{\alpha,\beta})\,\mathring{\pi}^\mu$, we have also

$$\begin{aligned} d\mathring{\mathfrak{F}} = \sum_{\mu,\beta=1}^4 &\left(\mathring{\xi}_5(\mathring{\mathcal{A}}_{\mu,\beta}) + 2\sum_{\alpha=1}^4 \mathring{\mathcal{A}}_{\alpha,\beta}\left[\mathring{\mathcal{A}}_\mu^\alpha + \Theta\,\delta_\mu^\alpha\right]\right)\mathring{\pi}^5 \wedge \mathring{\pi}^\mu \wedge \mathring{\pi}^\beta \\ &+ \sum_{\alpha,\beta,\mu=1}^4 \left(\mathring{\xi}_\mu(\mathring{\mathcal{A}}_{\alpha,\beta}) + 2\,q_\mu\,\mathring{\mathcal{A}}_{\alpha,\beta}\right)\mathring{\pi}^\mu \wedge \mathring{\pi}^\alpha \wedge \mathring{\pi}^\beta\,. \end{aligned} \tag{9.76}$$

Now, we want $d\mathring{\mathfrak{F}}$ to be a horizontal cotensor as $\mathring{\mathfrak{F}}$, *i.e.*, $i_{\hat{\xi}}\,\mathring{\mathfrak{F}} = 0$ and $i_{\hat{\xi}}\,d\mathring{\mathfrak{F}} = 0$. Therefore, the coefficient of the first term with $\mathring{\pi}^5$ in (9.76) must vanish, *i.e.*, the relation

$$\mathring{\xi}_5(\mathring{\mathcal{A}}_{\alpha,\beta}) + 2\,\Theta\,\mathring{\mathcal{A}}_{\alpha,\beta} = \sum_{\mu=1}^4 \left(\mathring{\mathcal{A}}_{\beta,\mu}\,\mathring{\mathcal{A}}_\alpha^\mu - \mathring{\mathcal{A}}_{\alpha,\mu}\,\mathring{\mathcal{A}}_\beta^\mu\right) \tag{9.77}$$

must hold. This formula can be more precisely interpreted when it is compared with another formula which will be deduced when we impose, in the sequel, the connection $\mathring{\omega}$ to be a *normal* projective Cartan connection.

Now, if $d\mathring{\mathfrak{F}} = 0$, then, necessarily, the first term with $\mathring{\pi}^5$ in the formula (9.76) vanishes but also the second one. Then, we obtain the following system of equations ($q_\alpha = 1$, $\alpha = 1, \ldots, 4$):

$$\mathring{\xi}_\mu(\mathring{\mathcal{A}}_{\alpha,\beta}) + \mathring{\xi}_\alpha(\mathring{\mathcal{A}}_{\beta,\mu}) + \mathring{\xi}_\beta(\mathring{\mathcal{A}}_{\mu,\alpha}) + 2\left[q_\mu\,\mathring{\mathcal{A}}_{\alpha,\beta} + q_\alpha\,\mathring{\mathcal{A}}_{\beta,\mu} + q_\beta\,\mathring{\mathcal{A}}_{\mu,\alpha}\right] = 0\,. \tag{9.78}$$

This system is no more no less than the system of *Maxwell's equations* expressed in the *non-holonomic* coframe $\{\mathring{\pi}^1, \ldots \mathring{\pi}^4\}$. To recover the Maxwell's equations in a holonomic coframe, say $\{d\varsigma^1, \ldots, d\varsigma^4\}$ or any other coframe with exact 1-forms, we set 1) $\mathring{\mathfrak{F}} = \sum_{\alpha<\beta=1}^4 \mathring{\mathfrak{F}}_{\alpha,\beta}\,\mathring{\pi}^\alpha\wedge\mathring{\pi}^\beta = \sum_{\alpha<\beta=1}^4 \mathfrak{F}_{\alpha,\beta}\,d\varsigma^\alpha\wedge d\varsigma^\beta$, 2) we express the coefficients $\mathfrak{F}_{\alpha,\beta}$ as linear expressions of the coefficients $\mathring{\mathfrak{F}}_{\alpha,\beta}$, and then 3) we can easily deduce the Maxwell's equations in the usual form: $\partial_\mu(\mathcal{A}_{\alpha,\beta}) + \partial_\alpha(\mathcal{A}_{\beta,\mu}) + \partial_\beta(\mathcal{A}_{\mu,\alpha}) = 0$, where $\partial_\alpha \equiv \partial\ /\partial\varsigma^\alpha$. Obviously, we can have also the inverse process beginning with the $\mathcal{A}$'s to obtain the $\mathring{\mathcal{A}}$'s.

Besides, we obtain the differential equations of the trajectory of a charged test particle in an electromagnetic field when setting $k \equiv 2\,q/c$ where q is the charge of the particle and c the speed of light (the factor 2 is due to the relation $\mathring{\mathfrak{F}}^\alpha_\beta/2 = \mathring{\mathcal{A}}^\alpha_\beta$). And, finally, we have:

$$\frac{d\mathring{u}^\alpha}{d\kappa} + \sum_{\beta,\mu=1}^{4} \mathring{\Gamma}^\alpha_{\beta,\mu}\,\mathring{u}^\beta\,\mathring{u}^\mu = \frac{q}{c}\sum_{\mu=1}^{4}\mathring{\mathfrak{F}}^\alpha_\mu\,\mathring{u}^\mu\,, \tag{9.79}$$

where $i_{\mathring{u}}\,d\mathring{u}^\alpha \equiv d\mathring{u}^\alpha/d\kappa$.

3. *The projective curvature 2-form and the normal projective Cartan connection*

Let $\mathring{\mathcal{R}}$ be the 2-form such that

$$\mathring{\mathcal{R}} \equiv d\mathring{\Gamma} + \mathring{\Gamma}\wedge\mathring{\Gamma}\,, \tag{9.80}$$

then, the projective curvature 2-form $\mathring{\Omega}$ associated with $\mathring{\omega}$ is defined by the following formula:

$$\mathring{\Omega} \equiv \mathring{\mathcal{R}} - \hat{\pi}(\mathring{\mathcal{R}}\,.\,\hat{\xi})\mathbb{1} = \mathring{\mathcal{R}} - \mathring{\mathcal{R}}^5_5\,\mathbb{1}\,. \tag{9.81}$$

In indexed notation, we have in full generality:

$$\mathring{\Omega} = \sum_{\mathfrak{h},\mathfrak{u},\mathfrak{v},\mathfrak{w}=1}^{5} \mathring{\Omega}^{\mathfrak{h}}_{\mathfrak{u},\mathfrak{v}\mathfrak{w}}\;\mathring{\xi}_{\mathfrak{h}}\otimes\mathring{\pi}^{\mathfrak{u}}\otimes(\mathring{\pi}^{\mathfrak{v}}\wedge\mathring{\pi}^{\mathfrak{w}})\,, \tag{9.82}$$

or, equivalently,

$$\mathring{\Omega} = \sum_{\mathfrak{h},\mathfrak{u},\mathfrak{v},\mathfrak{w}=1}^{5} \mathring{\Omega}^{\mathfrak{h}}_{\mathfrak{u}}\, \mathring{\xi}_{\mathfrak{h}} \otimes \mathring{\pi}^{\mathfrak{u}}\,, \tag{9.83}$$

where

$$\mathring{\Omega}^{\mathfrak{h}}_{\mathfrak{u}} \equiv \sum_{\mathfrak{v},\mathfrak{w}=1}^{5} \mathring{\Omega}^{\mathfrak{h}}_{\mathfrak{u},\mathfrak{v}\mathfrak{w}}\; \mathring{\pi}^{\mathfrak{v}} \wedge \mathring{\pi}^{\mathfrak{w}}\,. \tag{9.84}$$

Then, the pre-projective Cartan connection $\mathring{\omega}$ is a *normal* projective Cartan connection if $\mathring{\Omega}$ satisfies the following set of conditions in addition to the precedent conditions 1, 2 and 3 for $\mathring{\omega}$:

4. $i_{\hat{\xi}}\,\mathring{\Omega} = 0$ (horizontality of $\mathring{\Omega}$),

5. $\mathring{\Omega}^{\alpha}_{5} = 0$ for all $\alpha = 1, \ldots, 4$ (*torsion-free* condition for $\mathring{\omega}$),

6. $Tr(\mathring{\Omega}) = 0\,,$

7. $\mathring{\Omega}ic_{\alpha,\beta} \equiv \sum_{\mu=1}^{4} \mathring{\Omega}^{\mu}_{\alpha,\mu\beta} = 0$, where $\alpha, \beta = 1, \ldots, 4$ and $\mathring{\Omega}ic$ is the projective Ricci tensor defined from $\mathring{\Omega}$.

The two last conditions define a normal projective Cartan connection $\mathring{\omega}$. We discuss first the consequences deduced from the condition 6.

a. ***The condition 6.*** We must have $Tr(\mathring{\Omega}) = Tr(\mathring{\mathcal{R}}) - 5\mathring{\mathcal{R}}^{5}_{5} = 0$ where $\mathring{\mathcal{R}}^{5}_{5} = d\mathring{\Gamma}^{5}_{5} + \sum_{\mu=1}^{4} \mathring{\Gamma}^{5}_{\mu} \wedge \mathring{\Gamma}^{\mu}_{5}$. But, we have $Tr(\mathring{\mathcal{R}}) = dTr(\mathring{\Gamma}) + Tr(\mathring{\Gamma} \wedge \mathring{\Gamma}) = 0 + 0 = 0$ because $Tr(\mathring{\Gamma}) = 5\mathring{\Gamma}^{5}_{5} = 1/2\, d(\ln|\hat{g}|)$ from the condition 2 and $\mathring{\Gamma} \wedge \mathring{\Gamma}$ is an anti-symmetric 2-form-valued matrix. Therefore, we must have $\mathring{\mathcal{R}}^{5}_{5} = 0$ which is equivalent to the important relation:

$$\sum_{\alpha=1}^{4} \mathring{\Gamma}^{5}_{\alpha} \wedge \mathring{\Gamma}^{\alpha}_{5} = 0\,. \tag{9.85}$$

And then, from now and throughout, we have

$$\mathring{\Omega} = d\mathring{\Gamma} + \mathring{\Gamma} \wedge \mathring{\Gamma}\,. \tag{9.86}$$

We deduce also that

$$\mathring{\Omega}^{5}_{5} = 0\,. \tag{9.87}$$

In addition, from (9.55b), we find also after a quite long computation that the $\mathring{\Omega}^5_\alpha$'s satisfy the following relations:

$$\sum_{\beta=1}^{4}\left(\delta^\beta_\alpha - v^\beta\left[\sum_{\mu=1}^{4}\mathring{g}_{\alpha\mu}\,v^\mu\right]\right)\mathring{\Omega}^5_\beta = \sum_{\beta,\mu=1}^{4}\mathring{g}_{\alpha\beta}\,v^\mu\,\mathring{\Omega}^\beta_\mu\,, \tag{9.88a}$$

$$\left(2-\hat{g}(\hat{\xi},\hat{\xi})\right)\sum_{\beta=1}^{4}v^\beta\,\mathring{\Omega}^5_\beta = \sum_{\alpha,\beta,\mu=1}^{4}\mathring{g}_{\alpha\mu}\,\mathring{\Omega}^\mu_\beta\,v^\alpha\,v^\beta\,. \tag{9.88b}$$

b. ***The condition 5.*** The *torsion-free* condition $\mathring{\Omega}^\alpha_5 = 0$ means that the Pfaff system $\mathring{S} \equiv \{\mathring{\Gamma}^1_5,\ldots,\mathring{\Gamma}^4_5\}$ is integrable since $\mathring{\Omega}^\alpha_5 = 0$ is equivalent to

$$d\mathring{\Gamma}^\alpha_5 = \sum_{\beta=1}^{4}\mathring{\Gamma}^\beta_5\wedge\mathring{\omega}^\alpha_\beta\,, \tag{9.89}$$

where $\mathring{\omega}^\alpha_\beta = \mathring{\Gamma}^\alpha_\beta - \mathring{\Gamma}^5_5\,\delta^\alpha_\beta$. Hence, in full generality, $\mathring{\omega}$ is not a $\mathring{S}$-complete projective Cartan connection. Also, as a consequence, each coefficient $\mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{v}}$ becomes a linear combination of the $\mathring{\Gamma}^\alpha_5$'s. Thus, if we know $\mathring{\mathfrak{F}}$ (to know the functions v^α), $\mathring{\Gamma}^5_5$ and the $\mathring{\Gamma}^\alpha_5$'s then we can deduce all the coefficients $\mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{v}}$, and therefore, the connection $\mathring{\omega}$.

c. ***The condition 4.*** This condition has an important consequence in the general definition of the tensor $\mathring{\mathcal{A}}^\alpha_\beta$ and, implicitly, on the properties of the Faraday tensor $\mathring{\mathfrak{F}}$ defined on the five-dimensional manifold $\widehat{\mathcal{M}}$. Indeed, from the relation (9.65a), *i.e.*, from the relation $\mathring{\Gamma}^\alpha_5 = \left(\sum_{\mu=1}^4 q_\mu\,v^\mu - 1\right)\mathring{\pi}^\alpha + \sum_{\mu=1}^4\mathring{\xi}_\mu(v^\alpha)\,\mathring{\pi}^\mu$, we obtain the following important result:

$$\mathring{\xi}_5(\mathring{\mathcal{A}}_{\alpha,\beta}) + 2\,\Theta\,\mathring{\mathcal{A}}_{\alpha,\beta} = -\sum_{\mu=1}^{4}\mathring{\mathcal{A}}_{\alpha,\mu}\,\mathring{\mathcal{A}}^\mu_\beta - \Theta^2\,\mathring{g}_{\alpha\beta}\,. \tag{9.90}$$

Indeed, we have $i_{\hat{\xi}}\,\mathring{\Gamma} = 0$ (condition 3.a) and we must have also $i_{\hat{\xi}}\,\mathring{\Omega} = 0$ and, in particular, $i_{\hat{\xi}}\,\mathring{\Omega}^\alpha_5 = 0$ or, equivalently, $i_{\hat{\xi}}\,d\mathring{\Gamma}^\alpha_5 = 0$. Hence, differentiating $\mathring{\Gamma}^\alpha_5$, we obtain successively the following:

$$\begin{aligned}
d\mathring{\Gamma}^\alpha_5 &= \sum_{\mu=1}^{4}q_\mu\,dv^\mu\wedge\mathring{\pi}^\alpha + \Theta\,d\mathring{\pi}^\alpha + \sum_{\beta=1}^{4}\left(d\mathring{\mathcal{A}}^\alpha_\beta\wedge\mathring{\pi}^\beta + \mathring{\mathcal{A}}^\alpha_\beta\,d\mathring{\pi}^\beta\right),\\
&= \sum_{\mu=1}^{4}q_\mu\,dv^\mu\wedge\mathring{\pi}^\alpha + \sum_{\beta=1}^{4}\left(d\mathring{\mathcal{A}}^\alpha_\beta\wedge\mathring{\pi}^\beta + \left[\mathring{\mathcal{A}}^\alpha_\beta + \Theta\,\delta^\alpha_\beta\right]d\mathring{\pi}^\beta\right),
\end{aligned}$$

and from (9.42d), we can write also that

$$d\mathring{\Gamma}^{\alpha}_{5} = \sum_{\beta=1}^{4}\left(\mathring{\xi}_5(\mathring{\mathcal{A}}^{\alpha}_{\beta}) + \sum_{\tau=1}^{4}\left\{\mathring{\mathcal{A}}^{\alpha}_{\tau} + \Theta\,\delta^{\alpha}_{\tau}\right\}\left\{\mathring{\mathcal{A}}^{\tau}_{\beta} + \Theta\,\delta^{\tau}_{\beta}\right\}\right)\mathring{\pi}^5\wedge\mathring{\pi}^{\beta}$$
$$+ \sum_{\mu,\beta=1}^{4}\left(\sum_{\rho=1}^{4} q_{\rho}\,\mathring{\mathcal{A}}^{\rho}_{\mu}\,\delta^{\alpha}_{\beta} + \mathring{\xi}_{\mu}(\mathring{\mathcal{A}}^{\alpha}_{\beta}) + \ q_{\mu}\left[\mathring{\mathcal{A}}^{\alpha}_{\beta} + \Theta\,\delta^{\alpha}_{\beta}\right]\right)\mathring{\pi}^{\mu}\wedge\mathring{\pi}^{\beta}\,.$$

Then, setting $i_{\mathring{\xi}}\, d\mathring{\Gamma}^{\alpha}_{5} = 0$, we find that

$$\mathring{\xi}_5(\mathring{\mathcal{A}}^{\alpha}_{\beta}) + \sum_{\tau=1}^{4}\left\{\mathring{\mathcal{A}}^{\alpha}_{\tau} + \Theta\,\delta^{\alpha}_{\tau}\right\}\left\{\mathring{\mathcal{A}}^{\tau}_{\beta} + \Theta\,\delta^{\tau}_{\beta}\right\} = 0\,. \tag{9.91}$$

But, we have also the relations $\mathring{\xi}_5(\mathring{\mathcal{A}}_{\alpha,\beta}) = \sum_{\mu=1}^{4}\mathring{g}_{\alpha\mu}\,\mathring{\mathcal{A}}^{\mu}_{\beta}$ because $\mathring{\xi}_5(\mathring{g}_{\mu\nu}) = 0$ from $i_{\mathring{\xi}}\, d\mathring{g}_{\mu\nu} = 0$. Thus, from these relations and expanding the expression above, we obtain the formulas (9.90).

Now, if we compare the formulas (9.77) and (9.90), we deduce that (9.77) and (9.90) are equivalent to the following system of PDEs:

$$\sum_{\mu,\nu=1}^{4}\mathring{g}_{\mu\nu}\,\mathring{\mathcal{A}}^{\mu}_{\beta}\,\mathring{\mathcal{A}}^{\nu}_{\alpha} = \Theta^2\,\mathring{g}_{\alpha\beta}\,, \tag{9.92a}$$

$$\mathring{\xi}_5(\mathring{\mathcal{A}}_{\alpha,\beta}) + 2\,\Theta\,\mathring{\mathcal{A}}_{\alpha,\beta} = 0\,. \tag{9.92b}$$

The PDEs (9.92b) express what must be the variations of $\mathring{\mathfrak{F}}$ along the vertical one-dimensional manifold foliating $\widehat{\mathcal{M}}$. These variations cannot be deduced from the Maxwell's equations and there are independent on. Thus, there are no conflicts between (9.92b) and the Maxwell's equations on the spacetime $\mathcal{M}$.

The relations (9.92a) are remarkable if we come back to the explicit definition of $\mathring{\mathcal{A}}^{\alpha}_{\beta}$ with respect to the variables v^{α}. Indeed, the relations (9.92a) can also be written as

$$\sum_{\mu,\nu=1}^{4}\mathring{g}_{\mu\nu}\,\mathring{\xi}_{\beta}(v^{\mu})\,\mathring{\xi}_{\alpha}(v^{\nu}) = \Theta^2\,\mathring{g}_{\alpha\beta}\,. \tag{9.93}$$

In this form, we see more clearly that the PDEs (9.92a) define a Lie group bundle $\mathcal{G}$ over $\mathcal{M}$ (and not a Lie groupoid) of smooth diffeomorphisms v from $\mathcal{M}$ to $\mathcal{M}$ (where $\mathcal{M}$ is considered as a particular embedded leaf of $\widehat{\mathcal{M}}$) such that the restricted metric $\mathring{g}_{\mathcal{M}}$ on $\mathcal{M}$ of $\mathring{g}$ is *punctually conformally equivariant* with respect to $\mathcal{G}$. More precisely, the diffeomorphisms v are such that

$$v : \varsigma \in \mathcal{M} \subset \widehat{\mathcal{M}} \longrightarrow (v^{\alpha}(\varsigma)) \in \mathcal{M} \subset \widehat{\mathcal{M}}\,, \tag{9.94}$$

and

$$v^*(\mathring{g}_{\mathcal{M}} \circ v^{-1}) = \Theta^2\, \mathring{g}_{\mathcal{M}}\,. \tag{9.95}$$

Besides, the horizontality of $\mathring{\Omega}$ means, equivalently, that $i_{\hat{\xi}}\, d\mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{v}} = 0$ for all $\mathfrak{u}, \mathfrak{v} = 1, \ldots, 5$ since we have already the relations $i_{\hat{\xi}}\, \mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{v}} = 0$ from the condition 3.a. Then, because the $\mathring{\Gamma}^{\mathfrak{u}}_{\mathfrak{v}}$ are linear combinations of the $\mathring{\Gamma}^{\alpha}_{5}$'s (condition 5), this condition 4 reduces to the restricted condition $i_{\hat{\xi}}\, d\mathring{\Gamma}^{\alpha}_{5} = 0$. But this condition is satisfied from the PDEs (9.92).

d. ***The condition 7.*** We have the following decomposition for $\mathring{\Omega}^{\alpha}_{\beta}$:

$$\mathring{\Omega}^{\alpha}_{\beta} = R^{\alpha}_{\beta} - K^{\alpha}_{\beta}\,, \tag{9.96}$$

where

$$R^{\alpha}_{\beta} = d\mathring{\Gamma}^{\alpha}_{\beta} + \sum_{\mu=1}^{4} \mathring{\Gamma}^{\alpha}_{\mu} \wedge \mathring{\Gamma}^{\mu}_{\beta}\,, \tag{9.97a}$$

$$K^{\alpha}_{\beta} = \mathring{\Gamma}^{5}_{\beta} \wedge \mathring{\Gamma}^{\alpha}_{5}\,. \tag{9.97b}$$

Hence, the condition $\mathring{\Omega}ic_{\alpha,\beta} \equiv \sum_{\mu=1}^{4} \mathring{\Omega}^{\mu}_{\alpha,\mu\beta} = 0$ involves that

$$Ric_{\alpha,\beta} = \sum_{\mu=1}^{4} K^{\mu}_{\alpha,\mu\beta}\,, \tag{9.98}$$

where Ric is the Ricci tensor defined from R on the pseudo-Riemannian spacetime manifold $\mathcal{M}$. We can note that $Ric_{\alpha,\beta} \neq Ric_{\beta,\alpha}$ because the coframe $\mathring{\mathcal{B}}^*$ is nonholonomic. More precisely, we obtain that

$$\begin{aligned} Ric_{\alpha,\beta} = \sum_{\beta,\sigma=1}^{4} \mathring{g}_{\alpha\sigma} \Big\{ & \mathring{\mathcal{A}}^{\sigma}_{\mu}\, \mathring{\Gamma}^{\mu}_{5,\beta} + (\mathring{\Gamma}^{\sigma}_{5,\beta} - \mathring{\mathcal{A}}^{\sigma}_{\beta})\, \mathring{\Gamma}^{\mu}_{5,\mu} \\ & - \mathring{\Gamma}^{\sigma}_{5,\mu}\, \mathring{\Gamma}^{\mu}_{5,\beta} + \sum_{\rho=1}^{4} v^{\rho} \left[\mathring{\Gamma}^{\sigma}_{\rho,\mu}\, \mathring{\Gamma}^{\mu}_{5,\beta} - \mathring{\Gamma}^{\sigma}_{\rho,\beta}\, \mathring{\Gamma}^{\mu}_{5,\mu} \right] \Big\}\,. \end{aligned} \tag{9.99}$$

Besides, from (9.65a), we have

$$\mathring{\Gamma}^{\alpha}_{5,\beta} = \Theta\, \delta^{\alpha}_{\beta} + \mathring{\mathcal{A}}^{\alpha}_{\beta}\,, \tag{9.100}$$

and therefore, $Ric_{\alpha,\beta}$ is also such that

$$Ric_{\alpha,\beta} = \sum_{\sigma=1}^{4} \mathring{g}_{\alpha\sigma} \left\{ \sum_{\rho=1}^{4} \left[\mathring{\Gamma}^{\sigma}_{\rho,\mu} \, \mathring{\mathcal{A}}^{\mu}_{\beta} + \mathring{\Gamma}^{\sigma}_{\rho,\beta} \left(\sum_{\mu=1}^{4} \mathring{\mathcal{A}}^{\mu}_{\mu} - 3\,\Theta \right) \right] v^{\rho} - \Theta \, \mathring{\mathcal{A}}^{\sigma}_{\beta} \right\} + \mathring{g}_{\alpha\beta} \, \Theta \left[3\,\Theta + \sum_{\mu=1}^{4} \mathring{\mathcal{A}}^{\mu}_{\mu} \right] . \quad (9.101)$$

Then, the scalar Riemannian curvature denoted by Sc is such that

$$Sc \equiv \sum_{\alpha,\beta=1}^{4} \breve{g}^{\alpha\beta} \, Ric_{\alpha,\beta} \,, \quad (9.102)$$

and thus, we have:

$$Sc = \sum_{\beta=1}^{4} \left\{ \sum_{\alpha,\mu=1}^{4} \mathring{\Gamma}^{\alpha}_{\beta,\mu} \, \mathring{\mathcal{A}}^{\mu}_{\alpha} + \left(\sum_{\alpha=1}^{4} \mathring{\Gamma}^{\alpha}_{\beta,\alpha} \right) \left(\sum_{\mu=1}^{4} \mathring{\mathcal{A}}^{\mu}_{\mu} - 3\,\Theta \right) \right\} v^{\beta} + 3\,\Theta \left(4\,\Theta + \sum_{\mu=1}^{4} \mathring{\mathcal{A}}^{\mu}_{\mu} \right) . \quad (9.103)$$

We can deduce now the Einstein tensor $\mathring{\mathcal{G}}_{\alpha,\beta} \equiv Ric_{\alpha,\beta} - \frac{1}{2} \, \mathring{g}_{\alpha\beta} \, Sc$. We obtain:

$$\begin{aligned} \mathring{\mathcal{G}}_{\alpha\beta} = \sum_{\sigma=1}^{4} \mathring{g}_{\alpha\sigma} & \left\{ \sum_{\rho=1}^{4} \left[\sum_{\mu=1}^{4} \mathring{\Gamma}^{\sigma}_{\rho,\mu} \, \mathring{\mathcal{A}}^{\mu}_{\beta} + \mathring{\Gamma}^{\sigma}_{\rho,\beta} \left(\sum_{\nu=1}^{4} \mathring{\mathcal{A}}^{\nu}_{\nu} - 3\,\Theta \right) \right] v^{\rho} - \Theta \, \mathring{\mathcal{A}}^{\sigma}_{\beta} \right\} \\ & - \frac{1}{2} \, \mathring{g}_{\alpha\beta} \left\{ 6\,\Theta^2 + \Theta \sum_{\mu=1}^{4} \mathring{\mathcal{A}}^{\mu}_{\mu} + \sum_{\rho=1}^{4} \left[\sum_{\sigma,\mu=1}^{4} \mathring{\Gamma}^{\sigma}_{\rho,\mu} \, \mathring{\mathcal{A}}^{\mu}_{\sigma} \right. \right. \\ & \left. \left. + \sum_{\sigma=1}^{4} \mathring{\Gamma}^{\sigma}_{\rho,\sigma} \left(\sum_{\mu=1}^{4} \mathring{\mathcal{A}}^{\mu}_{\mu} - 3\,\Theta \right) \right] v^{\rho} \right\} . \end{aligned} \quad (9.104)$$

4. The metric field from the physical tensors

In this section, we indicate the procedure to get the metric field $\mathring{g}$ from the physical tensors, namely, the Faraday tensor $\mathring{\mathfrak{F}}$ and the energy-momentum tensor $\mathring{\mathfrak{T}}$. Here below are the steps to follow successively:

i. From the components $\mathfrak{F}^{\alpha}_{\beta}$ of the Faraday tensor $\mathfrak{F}$ given in any holonomic basis, we deduce its components $\mathring{\mathfrak{F}}^{\alpha}_{\beta}$ in the bases $\mathring{\mathcal{B}}$ and $\mathring{\mathcal{B}}^*$, and then $\mathring{\mathcal{A}}^{\alpha}_{\beta} \equiv \mathring{\mathfrak{F}}^{\alpha}_{\beta}/2$.

ii. We deduce the function Θ from the relation (9.92b) recalling that $\mathring{\xi}_5(\mathring{\mathcal{A}}_{\alpha,\beta}) = \sum_{\mu=1}^4 \mathring{g}_{\alpha\mu}\, \mathring{\mathcal{A}}^{\mu}_{\beta}$ or, equivalently, that $\mathring{\xi}_5(\mathring{\mathcal{A}}^{\alpha}_{\beta}) = \sum_{\mu=1}^4 \breve{g}^{\alpha\mu}\, \mathring{\mathcal{A}}_{\mu,\beta}$. More precisely, Θ is then such that

$$\Theta \equiv -\frac{1}{2}\,\mathring{\xi}_5\big(\ln(|\mathring{\mathcal{A}}^{\alpha}_{\beta}|)\big)\,, \tag{9.105}$$

for all $\alpha, \beta = 1, \ldots, 4$. The result must be independent on the indices α and β.

iii. Then, we deduce $\mathring{\Gamma}^{\alpha}_{5,\beta}$ from its definition (9.100), *i.e.*, $\mathring{\Gamma}^{\alpha}_{5,\beta} = \Theta\,\delta^{\alpha}_{\beta} + \mathring{\mathcal{A}}^{\alpha}_{\beta}$.

iv. We solve the system of PDEs

$$\mathring{\xi}_{\alpha}(v^{\beta}) = \mathring{\mathcal{A}}^{\beta}_{\alpha} \tag{9.106}$$

to get the functions v^{α} up to additive constants constrained by the relation $\Theta = \sum_{\mu=1}^4 q_{\mu}\, v^{\mu} - 1$.

v. We consider the Frobenius relations (9.89) for the $\mathring{\Gamma}^{\alpha}_5$'s:

$$d\mathring{\Gamma}^{\alpha}_5 = \sum_{\beta=1}^{4} \mathring{\Gamma}^{\beta}_5 \wedge \mathring{\omega}^{\alpha}_{\beta}\,, \tag{9.107}$$

where $\mathring{\omega}^{\alpha}_{\beta} = \mathring{\Gamma}^{\alpha}_{\beta} - \mathring{\Gamma}^{5}_{5}\,\delta^{\alpha}_{\beta}$. The coefficient $\mathring{\Gamma}^5_5$ is well-defined from the relation (9.65b): $\mathring{\Gamma}^5_5 = -\sum_{\alpha=1}^4 q_{\alpha}\,\mathring{\pi}^{\alpha}$, independently on any metric field $\mathring{g}$ or physical tensors such as the Faraday tensor. Then, the Pfaff system $\{\mathring{\Gamma}^1_5, \ldots, \mathring{\Gamma}^4_5\}$ constitutes a basis for the whole of the horizontal forms. Hence, there exists functions $\lambda^{\alpha}_{\beta,\mu}$ such that $\mathring{\Gamma}^{\alpha}_{\beta} = \sum_{\mu=1}^4 \lambda^{\alpha}_{\beta,\mu}\,\mathring{\Gamma}^{\mu}_5$ and $\lambda^{\alpha}_{\beta,\mu} = -\lambda^{\alpha}_{\mu,\beta}$. Then, knowing $\mathring{\Gamma}^5_5$ and the $\mathring{\Gamma}^{\alpha}_5$'s, we can deduce the functions $\lambda^{\alpha}_{\beta,\mu}$ from the Frobenius relations above, and then, the 1-forms $\mathring{\Gamma}^{\alpha}_{\beta}$.

vi. Given the energy-momentum tensor $\mathring{\mathfrak{T}}$ and solving the Einstein's equations $\mathring{\mathcal{G}}_{\alpha,\beta} = \mathring{\mathfrak{T}}_{\alpha,\beta}$, we deduce algebraically the coefficients $\mathring{g}_{\alpha,\beta}$ of the metric $\hat{g}$ in the cobasis $\mathring{\mathcal{B}}^*$. And then, the final metric $\tilde{g}$ is deduced up to a constant conformal factor.

Lastly, we obtain g up to a constant conformal factor. Thus, we have obtained the following.

Theorem 11. *Let $\mathcal{M}$ be a pseudo-Riemannian, simply connected, connected and non-closed manifold of class C^1 modeled on $\mathbb{R}P^4$ and g a metric field satisfying the Einstein's equations on $\mathcal{M}$. Then, given the Faraday tensor on $\mathcal{M}$, g is unique up to a constant conformal factor.*

Proof. Note that $\widehat{\mathcal{M}}$ must be of class C^2 and $\mathcal{M}$ non-closed from the foliation conditions. The conditions of simply connectedness and connectedness are need to ensure the unicity of the functions $\lambda^{\alpha}_{\beta,\gamma}$ over $\mathcal{M}$ which are solutions of linear algebraic equations. Indeed, we have a Galois covering [God71] from the "*continuity of roots property*" [Cos00, BCR98] and from the finite number of solutions of the system of algebraic equations for the $\lambda^{\alpha}_{\beta,\gamma}$. And then, we have, in fact, only one solution because of the universal covering due to the simple connexity and connexity of $\mathcal{M}$. □

X. CONCLUSION AND INTERPRETATIONS

In this section, we give a few interpretations on three aspects from the different geometrical tensors we obtain from the projective geometry yielded by relativistic positioning, framing and location systems. Firstly, we recall the fundamental assumptions we made. There are basically three:

1. The spacetime $\mathcal{M}$ is described by a Riemannian manifold of dimension 4 endowed with a Lorentzian metric; only implicitly time oriented, and thus, only time orientable.

2. There are propagating fields, meaning that no signals can travel in $\mathcal{M}$ at infinite velocity. Only a finite velocity is necessary. The latter can vary locally or not, be isotropic or not. The values for such velocities are considered as void of meaning intrinsically. These fields must only be such that encryption of coded informations is possible. We call such fields, '*fields of data*,' and, in particular, '*fields of coordinates*.' They are, somewhat, "hybrids" linking geometry and physics to information.[41] Nevertheless, we must assume there are no caustics at a first step.

3. The clocks of the emitting satellites of the constellation of a given relativistic positioning system are only generators of events of which the increasing identity numbers are broadcast in all directions of space. There do not measure a "time" as reading heads do when

[41] Somehow, such fields of data link the 'reality' to 'logos', breaking the so-called 'bifurcation', a notion introduced by A. N. Whitehead in his criticism of the Plato and Aristotle philosophies [Whi64]. It involves what we can call a 'semiophysics;' a contraction of semiotics and physics.

reading time data which would be encrypted or recorded on a substrate spacetime even if a time orientation is given on $\mathcal{M}$. As a consequence, the spacetime structure must be insensitive to scalings (more generally affine changes) along each worldline of each emitting satellite. We call this insensitivity the 'scale underdeterminacy' constraining any relativistic positioning system and at the ground of the underlying projective geometry of the spacetime described by these positioning systems.

Secondly, three interpretations can be made on the following geometrical/physical "objects:"

1. the Weyl's length connection,
2. the Einstein's equations and the Maxwell's equations.

The Weyl's length connection. In the present projective framework, the Weyl's length connection is, actually, defined from the embedding of $\mathcal{M}$ into $\widehat{\mathcal{M}}$. More precisely, we consider, firstly, a light-like curve $\gamma(\lambda) \subset \widehat{\mathcal{M}}$ with tangent light-like vector $\hat{k} \equiv (k^{\mathfrak{s}})$ such that $k^\alpha \equiv \frac{d\varsigma^\alpha}{d\lambda} \equiv \dot{\varsigma}^\alpha$ and $k^5 = 1$ in the nonholonomic basis $\widehat{\mathcal{B}}$, and thus, such that $\hat{k} \equiv \sum_{\mathfrak{h}=1}^{5} k^{\mathfrak{h}}\,\hat{\xi}_{\mathfrak{h}}$. Secondly, we consider also $\hat{k}$ to be horizontal, *i.e.*, $\hat{\pi}(\hat{k}) = 0 \iff i_{\hat{k}}\, d\varsigma^0 = 0$. Therefore, all along the curve γ, the relation

$$d\varsigma^5 = -\left(\sum_{\alpha=1}^{4} k^\alpha\right) d\lambda \tag{10.1}$$

holds. Furthermore, carrying over this relation within the context of the relativistic location process, if $\gamma(\lambda = 0)$ is the *anchor* event $a \equiv \gamma(\lambda = 0)$ of a localized event $e \equiv \gamma(\lambda = 1)$, then we deduce that

$$\varsigma^5 = -\int_0^1 \left(\sum_{\alpha=1}^{4} k^\alpha\right) d\lambda\,. \tag{10.2}$$

This relation defines the embedding of $\mathcal{M}$ into $\widehat{\mathcal{M}}$. Also, ς^5 is independent on the curve γ whatever is its type since the 1-form $\left(\sum_{\alpha=1}^{4} k^\alpha\right) d\lambda \equiv \sum_{\alpha=1}^{4} d\varsigma^\alpha$ is exact.

Moreover, this value for ς^5 is unique if 1) only one null geodesic exists from any anchor a to its corresponding localized event e, and 2) γ is a null geodesic. The former condition is always satisfied from the causal axiomatics based on the existence of unique message functions. Then, taken γ to be the unique null geodesic from a to e, we ascribe to e the four time stamps ς^α and the fifth one ς^5 determined by the integral (10.2). Then, considering that the two conditions above are satisfied, to each localized event e corresponds a unique point $\hat{p}_e$ in a five-dimensional grid of localization such that $\hat{p}_e \equiv (\varsigma^{\mathfrak{e}})$. Nevertheless, we can suggest that ς^5 is also related to a notion of variation of energy since this is the only isotropic, physical and non-geometric observable featuring the variation in a curved spacetime of a light ray along its pathway.

In addition, this means that ς^5 is necessary for localization (of events) but not for positioning (of users), and that the various tensors must be restricted on any worldline by setting the relation (10.2). Then, we see that the integrand in (10.2) can be ascribed to the Weyl's length connection ϕ such that

$$\phi \equiv -\sum_{\alpha=1}^{4} d\varsigma^\alpha \equiv d\varsigma^5\,. \tag{10.3}$$

With this interpretation, the Weyl's length connection is not path-dependent and it is associated with the length of an interval on a light-like path because any variation of ς^5 is due to propagation of signals emitted by the fifth satellite. Hence, the Einstein's criticism against this Weyl's length connection cannot be retained with such interpretation. Moreover, we can say that the Weyl geometry is a projective geometry coming from a five-dimensional manifold.

The Einstein's equations and the Maxwell's equations. É. Cartan proved that the torsion-free condition for the projective Cartan connection $\mathring{\omega}$ allows to determine completely and univocally the horizontal part $(\mathring{\omega}^\alpha_\beta)$ of $\mathring{\omega}$ once the projective geodesics are given. The latter can be obtained from four given polynomials P_α of degrees two with respect to the four variables $u^\alpha \equiv d\varsigma^\alpha/d\varsigma^4$ and with coefficients depending only on the five variables $\varsigma^{\mathfrak{h}}$. Thus, the torsion-free condition is not sufficient to completely and univocally determine the remaining 1-forms $\mathring{\omega}^5_\alpha$, *i.e.*, the soldering forms. As noticed in Remark 13, the condition of normality imposed on a torsion-free projective Cartan connection $\mathring{\omega}$ involves that once the soldering forms $\mathring{\omega}^5_\alpha$ are specified then the projective geodesics (the polynomials P_α) are univocally defined, and reciprocally. The condition of normality is expressed by the constraints on the connection $\mathring{\Gamma}$. Conversely, given soldering forms and projective geodesics determine completely and univocally a normal projective Cartan connection. But, the notion of geodesics can also be defined by metrics rather than by directly four polynomials P_α. Hence, É. Cartan chose the soldering forms as those defining metrics compatible with the projective connection and thus defining also the projective geodesics; and then, the unique normal projective connection.

But then, what is the role of the Einstein's equations and the Maxwell's equations in this projective framework? Actually, we find that the connection $\mathring{\omega}$ is completely defined once the energy-momentum tensor $\mathring{\mathfrak{T}}$ and the Faraday tensor $\mathring{\mathfrak{F}}$ are known. These tensors can be deduced from physical local observations without the need for any explicit knowledge of the metric field $\mathring{g}$. We need only a frame holonomic or not and a relativistic location system to evaluate, in this given frame, the coefficients of these fundamental physical tensors.

Appendices

Appendix A: The Marzke-Wheeler protocol

We recall first that the Marzke-Wheeler protocol [MW64] allows to Ehlers, Pirani and Schild [EPS72] and [Woo73] to define a sort of "scalar potential of metric" from which derives the "metric tensor" itself. The starting point, if we refer to the FIGURE A.1 for the explainations,

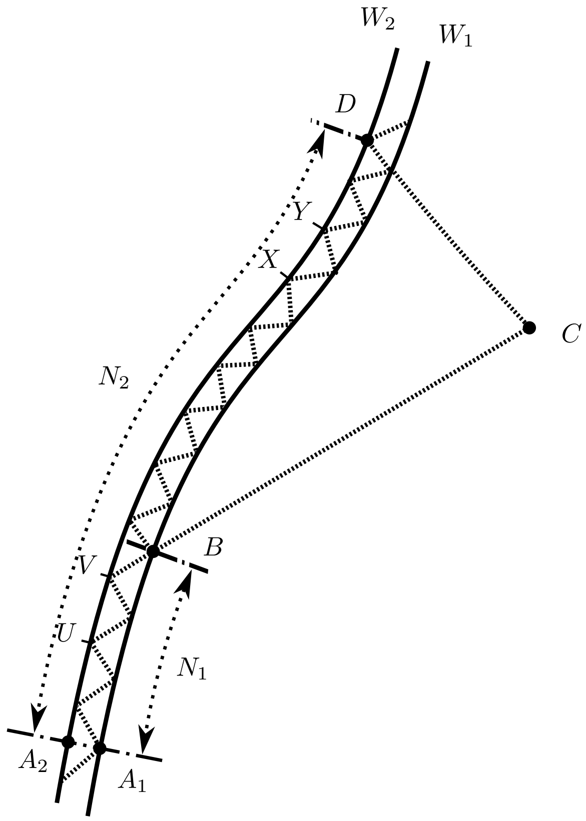


Figure A.1. Protocole de Marzke-Wheeler.

is the following: We have at disposal two worldlines W_1 et W_2. The worldline W_1 is a geodesic, and W_2 is a worldline of which the spatial distance to W_1 is constant and "small" (enventually infinitesimal). From the underlying spacetime geometry, W_2 can be also, possibly, a geodesic; but, in full generality, it is not really necessary.

Besides, a light echo is carried out, represented in the figure by a dashed zig-zag line between these two worldlines. Because of the assumed smallness of spatial distance between W_1 and W_2, certain intervals such as UV and XY on the worldline W_2 have very similar lengths. If W_1 is not a geodesic worldline, then the zig-zags would refer to an other field of light cones, and then, these two lengths could be extremely different. Actually, this protocol should be considered at the limit $W_2 \to W_1$ with W_1 geodesic.

In this situation, we can typically choose as units of the proper times shared by W_1 and W_2 all the lengths between two successive reflexions on W_1 or W_2; such as, for instance, UV and XY.

Now, let N_1 and N_2 be the numbers of round trips as indicated on the FIGURE A.1. Then, the pseudo-euclidean distance $d(A, B)$ between A and B is such that

$$d(A,B)^2 \equiv -\, c^2 \lim_{W_2 \to W_1} N_1\, N_2\,, \tag{A1}$$

where c is the speed of light. Indeed, going to the limit, and selecting a local chart (x, t) of the spacetime encompassing the FIGURE A.1, then, in this chart we can represent the "limit figure" by this other FIGURE A.2:

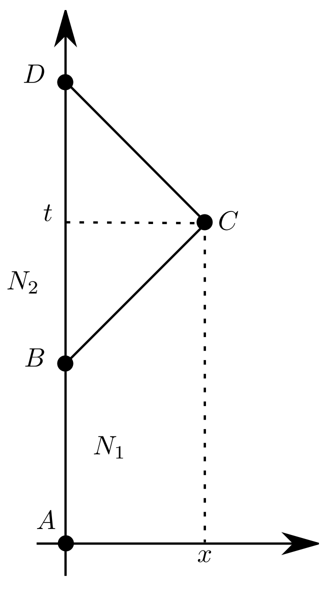


Figure A.2. The Marzke-Wheeler principle where A_1 and A_2 have been merged in A. Thus, we see that

$$c\,(N_2 - t) = c\,(t - N_1) = x\,.$$

Hence, we obtain that

$$c\,N_1 = c\,t - x\,, \qquad c\,N_2 = c\,t + x\,.$$

As a result, we have $-\,c^2\,N_1\,N_2 = x^2 - c^2\,t^2$; which constitutes well the canonical expression of the Lorentz metric.

Appendix B: Proofs of the two "factorization theorems"

1 Introduction

We denote by g the Lorentzian metric field defined on the spacetime and $d\tau^i$ the differential 1-form associated with the emission coordinate τ^i $(i = 1, \ldots, 4)$, then g can be written in the form $g \equiv \sum_{i,j=1}^{4} g_{ij}(\tau)\, d\tau^i\, d\tau^j$, and any basis vector of a $\{\ell\ell\ell\ell\}$-frame yielded by a RPS is the g-dual of a 1-form $d\tau^i$ rather than, merely, its dual vector $\frac{\partial}{\partial\tau^i}$. Also, g can be represented in the dual $\{\ell\ell\ell\ell\}$-frame $(\frac{\partial}{\partial\tau^1}, \ldots, \frac{\partial}{\partial\tau^4})$ by the matrix G such that

$$G \equiv \begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} \\ g_{21} & 0 & g_{23} & g_{24} \\ g_{31} & g_{32} & 0 & g_{34} \\ g_{41} & g_{42} & g_{43} & 0 \end{pmatrix}, \tag{B1}$$

where $g_{ij} = g_{ji} \neq 0$ if $i \neq j$ $(i,\, j = 1, \ldots, 4)$ and $\mathrm{sgn}(g_{ij}) = -\varepsilon$ whenever the signature of g is 2ε $(\varepsilon = \pm 1)$.

Coll and Pozo have made an extensive study of the algebraic properties of the class of metrics obtained specifically from RPSs [CP06]. They have shown, in particular, that the components g_{ij} can never been factorized in the general case, apart, possibly, at very particular events in the spacetime, *i.e.*, no set Λ of four nonvanishing functions ν_i exists in the general case such that, for instance, $g_{ij} \equiv \nu_i\,\nu_j$ for all $i,\, j = 1, \ldots, 4$ such that $i \neq j$. Obviously, this does not preclude, a priori, the existence of a particular system of emission coordinates such that the components of g factorize everywhere in the spacetime.

However, if $n \leqslant 4$, we show that there always exists a set Λ and a metric $\tilde{g}$ which is ℓ-*isometric* to g, in a meaning to be specified in the sequel, with factorized components, *i.e.*, there exists n nonvanishing functions $\tilde{\nu}_i$ such that $\tilde{g}_{ij} \equiv \tilde{\nu}_i \, \tilde{\nu}_j$ for all i, $j = 1, \ldots, n$ such that $i \neq j$. The number of nonvanishing components of g and $\tilde{g}$ is the same, *i.e.*, we have always $n(n-1)/2$ nonvanishing components out of the diagonal for either metric g and $\tilde{g}$. The essential difference between g and $\tilde{g}$ is that the $n(n-1)/2$ nonvanishing components of $\tilde{g}$ are not functionally independent.

This equivalence is obtained from a change of local dual null frame related to a local change of emission coordinates, and the ℓ-isometry between g and $\tilde{g}$ involves that the geometrical spacetime structure can be equivalently described by n functions only rather than by $n(n-1)/2$ functions.

In Minkowski spacetime, we can present the following example for the metrics g and $\tilde{g}$. We associate respectively with g and $\tilde{g}$ the matrices $G \equiv (g_{ij})$ and $\widetilde{G} \equiv (\tilde{g}_{ij})$, and we denote by $S \in SO(4, \mathbb{R})$ the Jacobian matrix associated with an affine change of emission coordinates. Then, with G and S such that

$$G = \begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}, \qquad S = \varepsilon \begin{pmatrix} \frac{1}{2}\Big(\frac{\sqrt{3}}{3} - 1\Big) & \frac{\sqrt{3}}{3} & 0 & \frac{1}{2}\Big(1 + \frac{\sqrt{3}}{3}\Big) \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \\ 0 & 0 & 1 & 0 \\ \frac{1}{2}\Big(1 + \frac{\sqrt{3}}{3}\Big) & \frac{\sqrt{3}}{3} & 0 & \frac{1}{2}\Big(\frac{\sqrt{3}}{3} - 1\Big) \end{pmatrix},$$

we obtain

$$\widetilde{G} = {}^T\!S \, G \, S = \begin{pmatrix} 0 & 1 & \frac{2\sqrt{3}}{3} & 2 \\ 1 & 0 & \frac{\sqrt{3}}{3} & 1 \\ \frac{2\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 & \frac{2\sqrt{3}}{3} \\ 2 & 1 & \frac{2\sqrt{3}}{3} & 0 \end{pmatrix},$$

where ${}^T\!S \, (\equiv \; S^{-1})$ is the transpose of S and $\varepsilon = \pm 1$. From, we note that no constants ν_i exist such that $g_{ij} \equiv \nu_i \, \nu_j$ for all i, $j = 1, \ldots, 4$ such that $i \neq j$. Indeed, there should be, in particular, the relation $g_{12}\, g_{13}/g_{23} = g_{12}\, g_{14}/g_{24} = (\nu_1)^2$. But the latter is not satisfied because $g_{12}\, g_{13}/g_{23} = 1$ and $g_{12}\, g_{14}/g_{24} = 2$. On the contrary, it is easy to show that $\tilde{g}_{ij} \equiv \tilde{\nu}_i\, \tilde{\nu}_j$ for all $i,\, j = 1, \ldots, 4$ such that $i \neq j$ whenever $\tilde{\nu}_1 = \sqrt{2}$, $\tilde{\nu}_2 = \sqrt{2}/2$, $\tilde{\nu}_3 = \sqrt{6}/3$ and $\tilde{\nu}_4 = \sqrt{2}$.

In this example, the change of emission coordinates is essentially a *punctual* algebraic issue, *i.e.*, an issue which is the same whatever is the given spacetime event where G is evaluated, because it is quite easy to find algebraically a matrix S which is the Jacobian matrix of a change of emission coordinates. In more general situations, we must solve systems of PDEs to connect a change-of-basis matrix to the Jacobian matrix of a change of emission coordinates. The two theorems we present below in Sec. B 2 and B 3 on the resolutions of these systems of PDEs can be considered independently of the physical application to RPSs although their interests in general relativity might be strongly relevant in complement to the theory of RPSs. In Sec. B 2, assuming $n \leqslant 4$, we prove the systematic existence of $\tilde{g}$ whatever is the Lorentzian metric field g, and in Sec. B 3, we prove the unicity of $\tilde{g}$ if we consider only orthogonal transformations between g and $\tilde{g}$. Also, the mathematical methods or tools employed in the proofs of these two theorems are, for instance, exhaustively indicated in [BCG+91]. These methods can be gathered under the designation of formal theory on the integrability of PDEs.

2 The equivalent generic metric fields

Let $\mathcal{M}$ be a smooth connected n-dimensional pseudo-Riemannian manifold endowed with a Lorentzian metric g represented as in (B1) in a given $\{\ell\ell\ell\ell\}$-frame defined on an emission coordinates chart $(U, \tau^1, \ldots, \tau^n)$ where the open $U \subset \mathcal{M}$. We denote by ∂_i the partial derivative with respect to the i-th emission coordinate τ^i of τ. Then, the present paper is devoted to the proof of the following result:

Theorem B.1. (First factorization theorem). *If $n \leqslant 4$, there always exists a smooth local diffeomorphism $f(\tau) = \psi$ and n smooth positive functions $\nu_i(\tau)$, both defined on an open neighborhood $V \subset U$ of any given point of U, such that for all $\tau \equiv (\tau^1, \ldots, \tau^n) \in V$ the relations*

$$\tilde{g}_{ij} \equiv \sum_{r,\, s=1}^{n} g_{rs}(f)(\partial_i f^r)(\partial_j f^s) = \epsilon_{ij}\, \nu_i\, \nu_j\,, \qquad i, j = 1, \ldots, n\,, \tag{B2}$$

hold with $\epsilon_{ij} = \mathrm{sgn}(g_{ij}) = \mathrm{sgn}(\tilde{g}_{ij})$ *whenever* $i \neq j$ *and* $\epsilon_{ij} = 0$ *otherwise. Then, we say that the "generic" metric* $\tilde{g}$ *is* ℓ-equivalent *to* g *(through* f*).*

Note that the off-diagonal terms of g are always nonvanishing. Furthermore, if $n \leqslant 3$, the result is trivial: take the identity map for f and the functions ν_i are unique. Moreover, if $n = 2$, we can make a separation of variables [CFM06b, CFM06a] in g_{12} such that each function ν_i $(i = 1, 2)$ depends on only one emission coordinate (because any two-dimensional Riemann manifold is conformally flat). In cases of dimension greater than 4, some constraints on the definition of g must be imposed.

The proof of this theorem presented below is made in the framework of the smooth category rather than the analytic category which is the standard situation for the application of the Cartan-Kähler theorem. Hence, no particular analytical criteria are discussed in relation to analytical boundary conditions for instance, and only the smooth Frobenius conditions are applied to check the integrability of the different PDEs involved in the proofs. Actually, we do not use either the Cartan-Kähler theorem or Cartan's test for involutivity. Hence, neither the computations of the codimensions of the polar spaces associated with the integral elements of certain flags nor the evaluations of their Kähler-regularities or regularities are performed [BCG⁺91]. The main reason is due to the non-standard way we "transform" a given set of algebraic equations defined in a jet bundle and associated with a system of PDEs to an associated Pfaff system of contact 1-forms. We just make a little step aside in the definition of this "transformation" with strong advantages in the proof as a result, as there appear to be some often unnoticed forms of indetermination in the definition of the associated Pfaff system of a system of PDEs.

a The systems of equations

Let π_n be the trivial fibration $\pi_n : \mathcal{M}^2 \equiv \mathcal{M} \times \mathcal{M} \longrightarrow \mathcal{M}$, corresponding to the projection onto the first factor. We denote by $J_k(\pi_n)$ the fiber bundle of jets of order $k \geqslant 0$ of the local smooth sections of π_n. In particular, we have $J_0(\pi_n) \equiv \mathcal{M}^2$ with local coordinates $(\tau, \psi) \equiv$

$(\tau^1, \ldots, \tau^n, \psi^1, \ldots, \psi^n)$ where τ is the *source* and ψ is the *target*. Furthermore, let

$$\psi_1 \equiv (\tau^1, \ldots, \tau^n, \psi^1, \ldots, \psi^n, \psi^1_1, \psi^1_2, \ldots, \psi^i_j, \ldots, \psi^n_{n-1}, \psi^n_n)$$

be a local system of coordinates on $J_1(\pi_n)$. We denote also by $\Pi_k(\pi_n) \subset J_k(\pi_n)$ the set of invertible elements of $J_k(\pi_n)$, *i.e.*, the set of k-jets of local smooth diffeomorphisms on $\mathcal{M}$. $\Pi_k(\pi_n)$ is a groupoid with source map $\alpha_k : \Pi_k(\pi_n) \longrightarrow \mathcal{M}$ where $\mathcal{M}$ is the first factor of $\mathcal{M}^2$ and the target map $\beta_k : \Pi_k(\pi_n) \longrightarrow \mathcal{M}$ where we project onto the second factor. Also, we denote by $\mathit{\Pi}_k(\pi_n)$ the presheaf of germs of local smooth α_k-sections of $\Pi_k(\pi_n)$. Then, we consider any solution of the system of PDEs (B2) as a sub-manifold of $\Pi_1(\pi_n)$ transversal to the α_k-fibers and defined from the following system $\mathcal{R}_1$ of equations on the presheaf $\mathit{\Pi}_1(\pi_n)$:

$$\mathcal{R}_1 : \quad \sum_{r,\, s=1}^{n} g_{rs}(\psi)\, \psi^r_i\, \psi^s_j - \epsilon_{ij}\, \nu_i(\tau)\, \nu_j(\tau) = 0\,, \qquad i, j = 1, \ldots, n\,, \tag{B3}$$

where $\nu_i(\tau) > 0$.

Then, we denote also by $\hat{g}_{ij}$ the terms such that

$$\hat{g}_{ij} \equiv \sum_{r,\, s=1}^{n} g_{rs}(\psi)\, \psi^r_i\, \psi^s_j\,, \qquad i, j = 1, \ldots, n\,.$$

Hence, $\mathcal{R}_1$ is also the following set of algebraic equations:

$$\begin{cases} \hat{g}_{ii} = 0\,, \\ \hat{g}_{ij} = \epsilon_{ij}\, \nu_i(\tau)\, \nu_j(\tau)\,, \qquad i \neq j = 1, \ldots, n\,. \end{cases}$$

We deduce easily for all distinct indices i, j and k that $\epsilon_{ij}\, \epsilon_{jk}\, \hat{g}_{ij}\, \hat{g}_{jk} = (\nu_j)^2\, \hat{g}_{ik}\, \epsilon_{ik}$. Then, from $\mathcal{R}_1$, we must have the following equivalent system of equations:

$$\begin{cases} \hat{g}_{ii} = 0\,, \\ |\hat{g}_{ij}\, \hat{g}_{jk}| = (\nu_j)^2\, |\hat{g}_{ik}|\,, \quad \text{for all } i, j \text{ and } k \text{ distinct in } \{1, \ldots, n\}. \end{cases} \tag{B4}$$

In particular, if $n = 4$ in (B4), then, apart from the set of equations $\hat{g}_{ii} = 0$, the second set of equations are necessarily satisfied unless the two following deduced equations are not:

$$|\hat{g}_{12}\, \hat{g}_{34}| = |\hat{g}_{13}\, \hat{g}_{24}|\,, \qquad |\hat{g}_{13}\, \hat{g}_{24}| = |\hat{g}_{14}\, \hat{g}_{23}|\,.$$

Therefore, if $n = 4$, the system $\mathcal{R}_1$ reduces to the following set of PDEs:

$$\begin{cases} \hat{g}_{ii} = 0\,, \qquad i = 1, \ldots, 4\,, \\ |\hat{g}_{12}\,\hat{g}_{34}| = |\hat{g}_{13}\,\hat{g}_{24}| \neq 0\,, \\ |\hat{g}_{13}\,\hat{g}_{24}| = |\hat{g}_{14}\,\hat{g}_{23}| \neq 0\,. \end{cases}$$

Rewriting this system of PDEs without the absolute values, we obtain

$$\mathcal{R}'_1 : \quad \begin{cases} \mathcal{F}_i(\psi_1) \equiv \sum_{r,s=1}^4 g_{rs}(\psi)\,\psi_i^r\,\psi_i^s = 0\,, \qquad i = 1, \ldots, 4\,, \\ \mathcal{F}_5(\psi_1) \equiv \sum_{i,j,k,h=1}^4 g^{\epsilon}_{ijkh}(\psi)\,\psi_1^i\,\psi_2^j\,\psi_3^k\,\psi_4^h = 0\,, \\ \mathcal{F}_6(\psi_1) \equiv \sum_{i,j,k,h=1}^4 g^{\epsilon'}_{ijhk}(\psi)\,\psi_1^i\,\psi_2^j\,\psi_3^k\,\psi_4^h = 0\,, \end{cases} \tag{B5}$$

where

$$g^{\epsilon}_{ijkh} \equiv g_{ij}\,g_{kh} - \epsilon\,g_{ik}\,g_{jh}\,, \qquad \epsilon = \pm 1\,, \quad i, j, k, h = 1, \ldots, 4\,.$$

Actually, from the proof of the Lemma below, we must always take $\epsilon = \epsilon'$. Then, solving $\mathcal{R}'_1 \subset \mathit{\Pi}_1(\pi_4)$ is equivalent to solve $\mathcal{R}_1$.

Before going further, we must know if there exist solutions to the system of homogeneous polynomial equations (B5) in the variables ψ_j^i whenever ψ and τ are fixed. Actually, we obtain:

Lemma B.1. *Let E_1 be the system of algebraic equations with respect to the 16 variables ψ_j^i such that*

$$E_1 : \quad \sum_{r,\,s=1}^{4} g_{rs}(\psi)\,\psi_i^r\,\psi_j^s - \epsilon_{ij}\,\nu_i(\tau)\,\nu_j(\tau) = 0\,, \qquad i, j = 1, \ldots, 4\,.$$

Then, E_1 always admits an open set S_ψ^1 of real solutions ψ_j^i such that $\det(\psi_j^i) \neq 0$ whatever are the source τ and the target ψ fixed.

Proof. We consider that solving E_1 is equivalent to solve the system of algebraic equations E'_1 deduced from $\mathcal{R}'_1$ whenever ψ is fixed regardless of the values for τ. Then, first, we denote by ϕ_i the linearly independent column vectors such that $\phi_i \equiv (\psi_i^j)$. From the first four equations $\mathcal{F}_i(\psi_1) = 0$ $(i = 1, \ldots, :, 4)$, the vectors ϕ_i must be light-like vectors which still exist since

g is Lorentzian. Second, the last two functions can be rewritten as $\mathcal{F}_5(\psi_1) = g(\phi_1, \phi_5)$ and $\mathcal{F}_6(\psi_1) = g(\phi_1, \phi_6)$ where $\phi_5 \equiv \hat{g}_{34}\,\phi_2 - \epsilon\,\hat{g}_{24}\,\phi_3$ and $\phi_6 \equiv \hat{g}_{34}\,\phi_2 - \epsilon'\hat{g}_{23}\,\phi_4$. Therefore, the nonvanishing vectors ϕ_5 and ϕ_6 are collinear to ϕ_1 or time-like ($\hat{g}_{ij} \neq 0$ if $i \neq j$). However, because the four vectors ϕ_i ($i = 1, \ldots, 4$) are linearly independent, then ϕ_5 and ϕ_6 must be time-like. Hence, the signs of their norms $g(\phi_5,\, \phi_5)$ and $g(\phi_6,\, \phi_6)$ are equal to the sign of the signature 2ε of g, *i.e.*, we have

$$\mathrm{sgn}(\hat{g}_{34}\,\hat{g}_{24}\,\hat{g}_{23}) = -\epsilon\varepsilon = -\epsilon'\varepsilon\,.$$

Thus, in particular, we must have $\epsilon' = \epsilon$ in the system (B5). Besides, ϵ is arbitrary, and then, from now and throughout, we set also $\epsilon = \varepsilon$ (the proof would be strictly similar with $\epsilon = -\varepsilon$ permuting the words negative and positive in the text below; or the symbols $<$ and $>$ as well). As a result, we have solutions to the system E_1' if and only if [42]

$$\hat{g}_{34}\,\hat{g}_{24}\,\hat{g}_{23} < 0\,. \tag{B6}$$

Next, we consider the expression $\hat{g}_{34}\,\hat{g}_{24}\,\hat{g}_{23}$ as a quadratic form Q with respect to ϕ_2. We obtain $\hat{g}_{34}\,\hat{g}_{24}\,\hat{g}_{23} = Q(\phi_2,\, \phi_2) \equiv \sum_{i,j=1}^{4} Q_{ij}\,\phi_2^i\,\phi_2^j$ where

$$Q_{ij} = \left(\sum_{h,\,k=1}^{n} g_{rs}(\psi)\,\phi_3^h\,\phi_4^k\right)\left(\sum_{s=1}^{4} g_{js}(\psi)\,\phi_3^s\right)\left(\sum_{r=1}^{4} g_{ir}(\psi)\,\phi_4^r\right),$$

and then, the inequality (B6) is always satisfied if Q is not a positive elliptic form. For, it suffices that one of the diagonal terms Q_{ii} to be non-positive since, in this case, it implies the existence of basis vectors of non-positive norms with respect to Q if Q is non-degenerate.[43] We cannot ensure in full generality the non-degeneracy of Q, and thus, we impose, in particular, the condition $Q_{11} < 0$ only and not the condition $Q_{11} = 0$. Proceeding in the same way, the term Q_{11} is still considered as a quadratic form R with respect to ϕ_3. And again, we have $R(\phi_3, \phi_3) \equiv Q_{11} < 0$ if R is not a positive elliptic form. For the same reasons as above for Q_{11},

[42] Note that this inequality illustrates the first form of the Tarski-Seidenberg theorem [BCR98, Cos00].

[43] We can use also the Coll-Morales rules [CM93, see § III] generalizing more effectively the Jacobi, Gundelfinger and Frobenius rules with the notion of *causal sequence* $(i_1,\, i_2,\, i_3) \equiv (\mathrm{sgn}(\triangle_1),\, \mathrm{sgn}(\triangle_2),\, \delta\ \mathrm{sgn}(\triangle_3))$ where the $\triangle_k$'s are the first three *leading principal minors* of Q of order k and δ is the *determinant index*. In the present case, the causal sequence should differ from the causal sequence (1, 1, 1).

this condition is always satisfied, in particular, if there exists a diagonal term R_{ii} such that $R_{ii} < 0$. We consider the term R_{22} such that

$$R_{22} \equiv g_{12}(\psi) \left(\sum_{i=1}^{4} g_{1i}(\psi)\, \phi_4^i \right) \left(\sum_{j=1}^{4} g_{2j}(\psi)\, \phi_4^j \right). \tag{B7}$$

Then, because g is non-degenerate, the coefficients g_{2j} and g_{1j} for $j = 1, \ldots, 4$ cannot be simultaneously proportional with the same proportionality factor. Therefore, the two hyperplanes in $\mathbb{R}^4$ defined by the two last factors in (B7) and linear with respect to ϕ_4 are strictly distinct. As a result, $\mathbb{R}^4$ is divided by these two hyperplanes into four connected open subsets. Then, we can always find a vector $\phi_4 \in \mathbb{R}^4$ in one of these four subsets such that $\left(\sum_{i=1}^4 g_{1i}(\psi)\, \phi_4^i\right) \left(\sum_{j=1}^4 g_{2j}(\psi)\, \phi_4^j\right)$ has the opposite sign of $g_{12}(\psi)(\neq 0)$ and thus such that $R_{22} < 0$. Hence, there always exists an open set S_ψ^1 of real solutions to the system E_1' (and then E_1) whatever are the source τ and the target ψ. □

In addition to the previous Lemma, from the inequality (B6) and the 'continuity of roots' property[44] (see p. 363 of Ref. [Whi72]), we deduce that, given a point ψ, there always exists a maximal open subset $U_\psi \subset \mathcal{M}$ of ψ such that this set of solutions S_ψ^1 is always an open smooth manifold of *constant* dimension at least 10 on U_ψ. As a result, U_ψ is also necessarily closed, but then, because $\mathcal{M}$ is connected, we deduce that $\dim S_\psi^1 \equiv m$ is a constant on $\mathcal{M}$. Moreover, $\alpha_1 \times \beta_1$ is a surmersion on $\mathcal{M}^2$, and thus, the latter has no critical points in $\mathcal{R}_1'$. Therefore, we obtain that $m = 10$ (see Lemma 1, p. 11 of Ref. [Mil97]).

It follows that the restrictions to $\mathcal{R}_1'$ of the source and target maps are surmersions, and then, the system $\mathcal{R}_1'$ is, respectively, *formally integrable* (as a system of local diffeomorphisms defined on the whole of $\mathcal{M}$), and *homogeneous* (transitive diffeomorphisms from opens to any other opens in $\mathcal{M}$) [BCG+91]. And then, $\mathcal{R}_1'$ is a differentiable manifold such that $\dim \mathcal{R}_1' = 18$.

[44] The 'continuity of roots' property ensures the roots of a given finite set of algebraic equations to be continuously depending on the coefficients parameterizing these algebraic equations.

b The Pfaff systems and the proof of Theorem B.1

We consider the following canonical contact structure S_0 of width n (*i.e.*, n-flag [KR02, Mor04]) and length 1 on $\Pi_1(\pi_n)$ generated by the set $\{\omega^1, \omega^2, \ldots, \omega^n\}$ of contact 1-forms $\omega^i \in T^*J_1(\pi_n)$ such that

$$S_0 : \quad \begin{cases} \omega^1 = & d\psi^1 - \sum_{i=1}^n \psi^1_i \, d\tau^i \,, \\ \omega^2 = & d\psi^2 - \sum_{j=1}^n \psi^2_j \, d\tau^j \,, \\ \ldots = & \ldots\ldots\ldots\ldots\ldots\ldots \,, \\ \omega^n = & d\psi^n - \sum_{k=1}^n \psi^n_k \, d\tau^k \,. \end{cases} \tag{B8}$$

Obviously, the *terminal system* S_1 of S_0 is vanishing [KR02]. Then, we complement the set of contact 1-forms generating S_0 with another set of 1-forms ω^i_j on $\Pi_1(\pi_n)$ defined by the relations:

$$\omega^i_j \equiv d\psi^i_j - \sum_{k=1}^n z^i_{jk}(\psi_1) \, d\tau^k \,, \qquad\qquad i, j = 1, \ldots, n \,, \tag{B9}$$

where any given set of functions $z^i_{jk}(\psi_1) \in C^\infty(\Pi_1(\pi_n))$ (with $i, j, k, h = 1, \ldots, n$) must satisfy the relations

$$z^i_{jk}(\psi_1) = z^i_{kj}(\psi_1) \,, \qquad\qquad D_k z^i_{jh}(\psi_1) = D_h z^i_{jk}(\psi_1) \,, \tag{B10}$$

where D_k is the formal differentiation with respect to τ^k defined by the formula

$$D_k \equiv \frac{\partial}{\partial \tau^k} + \sum_{i=1}^n \psi^i_k \frac{\partial}{\partial \psi^i} + \sum_{i,j=1}^n z^i_{jk}(\psi_1) \frac{\partial}{\partial \psi^i_j} \,, \qquad\qquad k = 1, \ldots, n \,.$$

From this definition and for any smooth function $\mathcal{F}$ defined on $J_1(\pi_n)$ we find that the commutator $[D_k, D_h]$ satisfies the relation

$$[D_k, D_h](\mathcal{F}) = \sum_{i,j=1}^n \left(D_k z^i_{jh} - D_h z^i_{jk} \right) \frac{\partial \mathcal{F}}{\partial \psi^i_j} \,. \tag{B11}$$

Then, we set the following:

Definition B.1. *We denote by* $T_0(z) \supseteq S_0$ *the contact structure generated by the contact 1-forms* ω^i *and the 1-forms* ω^i_j $(i, j = 1, \ldots, n)$.

In particular, from (B11) and the relation $d^2\omega = 0$ for any smooth p-forms ω in $\Lambda T^* J_1(\pi_n)$, we deduce also that the *Martinet structure tensor* [Mar74] $\delta \equiv d \mod T_0(z)$ is such that $\delta^2 = 0$.

Moreover, from relations (B8) and (B9), we obtain:

$$\begin{cases} d\omega^i = \sum_{k=1}^n d\tau^k \wedge \omega^i_k \,, \\ d\omega^j_k = \sum_{h,r,s=1}^n \left(\dfrac{\partial z^j_{kh}}{\partial \psi^r_s}\right) d\tau^h \wedge \omega^r_s + \sum_{h,r=1}^n \left(\dfrac{\partial z^j_{kh}}{\partial \psi^r}\right) d\tau^h \wedge \omega^r \,, \end{cases}$$

and then, $T_0(z)$ satisfies the Frobenius conditions (equivalent to $\delta^2 = 0$) and is an integrable Pfaff system on $\Pi_1(\pi_n)$.

Next, we consider $\mathcal{R}'_1$ as a presheaf $\mathcal{I}_1$ of ideals locally finitely generated by the functions $\mathcal{F}_i$ $(i = 1, \ldots, 6)$ defined on $\Pi_k(\pi_4)$ and we assume that any manifold on which this presheaf vanishes, *i.e.*, the sub-manifold defined from a solution, is an integral sub-manifold of a contact structure $T_0(z)$ in $\Pi_k(\pi_4)$. Then, we set the following.

Definition B.2. *Given a set of functions z satisfying* (B10)*, we denote by $V_1(z)$ the foliation of the integral sub-manifolds in $\Pi_k(\pi_4)$ defined by the contact structure $T_0(z)$.*

This latter version conforms better with the classical concepts of integral manifolds and differs from the approach of PDEs translated in terms of presheafs of Pfaff systems of contact 1-forms satisfying the Frobenius conditions (see for instance Ref. [Bot70]).

As a consequence, denoting by $\mathcal{J}_1(z)$ the presheaf of differential ideals generated by $T_0(z)$ on $J_1(\pi_4)$, we say that

Definition B.3. *The system of PDEs $\mathcal{R}'_1$ is integrable on $\mathcal{M}$ if there exists a foliation $V_1(z)$ and a nonvanishing presheaf $\mathcal{J}_1(z)$ such that $\mathcal{I}_1 \subseteq \mathcal{J}_1(z)$ on $V_1(z)$.*

In other words, if a set of functions z^i_{jk} exists satisfying the latter condition, a smooth local diffeomorphism f of $\mathcal{M}$ is a solution of $\mathcal{R}'_1$ if and only if

$$\begin{cases} \mathcal{F}_i(j_1(f)) = \iota_0 \,, & i = 1, \ldots, 6 \,, \\ f^*(\omega^i) = 0 \,, & f^*(\omega^j_k) = 0 \,, \qquad i,j,k = 1, \ldots, 4 \,, \end{cases}$$

where ι_0 is the zero function on $\mathcal{M}$ and $j_1(f)$ is the first prolongation of f; and thus a local section of $\Pi_1(\pi_4)$. Hence, from (B9), we obtain that

$$\begin{cases} df^i = \sum_{k=1}^4 (\partial_k f^i)\, d\tau^k\,, & i = 1, \ldots, 4\,, \\ df^j_k = \sum_{h=1}^4 z^j_{kh}(j_1(f))\, d\tau^h\,, & j, k = 1, \ldots, 4\,. \end{cases}$$

And then, from the second order of derivation and from the successive prolongations, all of the derivatives of f are functionals of the derivatives of f of order less than or equal to one. As a result, a Taylor expansion for f can be deduced with Taylor coefficients defined from the Taylor coefficients of f of order less than or equal to one only. Thus, we obtain a formal Taylor expansion for f which can be convergent on a suitable relatively compact open neighborhood U_τ of any point $\tau \in \mathcal{M}$ if some Lipschitzian conditions on the functions z^j_{kh} are satisfied on $(\alpha_1)^{-1}(U_\tau) \cap V_1(z)$; justifying the definition of integrability given above for $\mathcal{R}'_1$. Then, from these preliminaries, the proof of Theorem B.1 is the following.

Proof of Theorem B.1. To satisfy the condition $\mathcal{I}_1 \subseteq \mathcal{J}_1(z)$ on $V_1(z)$, we must have $d\mathcal{F}_i \equiv 0$ mod $T_0(z)$ for all of the indices $i = 1, \ldots, 6$ on $V_1(z)$. We obtain the following system $\mathcal{S}(z)$ of 24 linear equations with 24 unknowns z^i_{jk}:

$$\delta\mathcal{F}_i = 0 \Longrightarrow \sum_{r,s=1}^{4} \left(\left(\sum_{k=1}^{4} (\partial_k g_{rs})(\psi)\, \psi^k_h \right) \psi^r_i\, \psi^s_i + g_{rs}(\psi)\, \psi^r_i\, z^s_{ih} \right) = 0\,,$$

$$\begin{aligned} \delta\mathcal{F}_5 = 0 \Longrightarrow \sum_{i,j,k,h,r=1}^{4} & (\partial_r g^\varepsilon_{ijkh})(\psi)\, \psi^i_1\, \psi^j_2\, \psi^k_3\, \psi^h_4\, \psi^r_s \\ & + \sum_{i,j,k,h=1}^{4} g^\varepsilon_{ijkh}(\psi) \left\{ \psi^i_1\, \psi^j_2\, \psi^k_3\, z^h_{4s} + \psi^i_1\, \psi^j_2\, z^k_{3s}\, \psi^h_4 \right. \\ & \left. + \psi^i_1\, z^j_{2s}\, \psi^k_3\, \psi^h_4 + z^i_{1s}\, \psi^j_2\, \psi^k_3\, \psi^h_4 \right\} = 0\,, \end{aligned}$$

$$\begin{aligned} \delta\mathcal{F}_6 = 0 \Longrightarrow \sum_{i,j,k,h,r=1}^{4} & (\partial_r g^\varepsilon_{ijhk})(\psi)\, \psi^i_1\, \psi^j_2\, \psi^k_3\, \psi^h_4\, \psi^r_s \\ & + \sum_{i,j,k,h=1}^{4} g^\varepsilon_{ijhk}(\psi) \left\{ \psi^i_1\, \psi^j_2\, \psi^k_3\, z^h_{4s} + \psi^i_1\, \psi^j_2\, z^k_{3s}\, \psi^h_4 \right. \\ & \left. + \psi^i_1\, z^j_{2s}\, \psi^k_3\, \psi^h_4 + z^i_{1s}\, \psi^j_2\, \psi^k_3\, \psi^h_4 \right\} = 0\,. \end{aligned}$$

Note that if $n > 4$ we have more equations than unknowns, and then not all metric fields g are admissible to satisfy the conditions of the theorem. Now, setting for all of the functions z^i_{jk} the relations

$$z^i_{jk}(\psi_1) = \psi^i_j \sum_{h=1}^{4} \psi^h_k \, z_h(\psi)\,, \tag{B12}$$

where the functions z_k depend on ψ, we find that the unique solution of $\mathcal{S}(z)$ ($\det \mathcal{S}(z) = 0$ on a closed subset C of matrices ψ_1 whenever the source τ and the target ψ are fixed, but, because S^1_ψ is open, we can always find matrices $\psi_1 \notin C$ such that $\det \mathcal{S}(z)$ is nonvanishing) is the set of functions z^i_{jk} such that

$$z_k(\psi) = -\frac{1}{8} \sum_{i,j=1}^{4} g^{ij}(\psi)\,(\partial_k g_{ij})(\psi) \equiv -\frac{1}{4} \sum_{i=1}^{4} \Gamma^i_{ik}(\psi)\,, \tag{B13}$$

where the Γ^i_{jk} are the Christoffel symbols of g. Then, it remains to see that the conditions (B10) are satisfied. For, we must have the relations

$$\sum_{h=1}^{4} \left(z^i_{jr}(\psi_1)\,\psi^h_k - z^i_{jk}(\psi_1)\,\psi^h_r \right) z_h(\psi) = 0\,,$$

which are, actually, verified with the functions z^i_{jk} given by the relations (B12) with (B13). Moreover, because no algebraic constraints exist on ψ_1, apart from those obtained from the vanishing of the functions $\mathcal{F}_i$ which are elements of $\mathcal{I}_1 \subseteq \mathcal{J}_1(z)$, then $V_1(z)$ foliates the whole of the open manifold $\Pi_1(\pi_4)$. Furthermore, the 1-forms ω^k and ω^i_j are the so-called *basic 1-forms* [Mol77] associated with any *complete transversally parallelizable* foliation. Lastly, at any given point τ and from the Lemma, the finite system of equations (B5) in the variables ψ_1 always have solutions, and the set of positive functions ν_i is not unique. □

Besides, we note that $\mathcal{R}_1$ is not a Lie groupoid [Mac87] because if g is ℓ-equivalent to $\epsilon_{ij}\,\nu_i\,\nu_j$ and $\epsilon_{ij}\,\tilde\nu_i\,\tilde\nu_j$ through, respectively, the diffeomorphisms f and $\tilde f$, then, there may not always be four positive functions $\hat\nu_i$ such that g would be ℓ-equivalent to $\epsilon_{ij}\,\hat\nu_i\,\hat\nu_j$ through $f \circ \tilde f$ or $\tilde f \circ f$. Nevertheless, we have an associated *principal groupoid* regarded as the graph of the ℓ-equivalence, and then the equivalence class $[g]$ of the given metric g is a source fiber in this groupoid.

3 The isometric equivalence

If $\mathcal{M}$ is time oriented, *i.e.*, there exists a complete (future time-like) vector field ξ on $\mathcal{M}$, then, because $\mathcal{R}_1$ is also a differentiable α_1-fiber bundle, we also have in the smooth category the following:

Theorem B.2. (Second factorization theorem). *If $n = 4$, then, given a Lorentzian metric g on $\mathcal{M}$ assumed to be time oriented, connected and simply connected, then, there exists only one smooth diffeomorphism $f^i(\tau) \equiv \psi^i$ being a solution of $\mathcal{R}_1$ of which the Jacobian matrix is an element of $O(4,\mathbb{R})$; and, as a result, there is a unique set of four positive functions ν_i. Also, the unique ℓ-equivalent metric field $\tilde{g}$ is* 'isometrically equivalent' *to g. Then $\tilde{g}$ is said to be* ℓ-isometric *to g and* ℓ-generic.

Proof. Let $\psi_0 \equiv (\tau, \psi)$ be any point in $\mathcal{M}^2$ and a matrix $\Psi \equiv (\psi^i_j) \in \alpha_1^{-1}(\tau) \times \beta_1^{-1}(\psi) \in \mathcal{R}_1^{\psi_0} \equiv \{\Psi \,/\, (\tau,\, \psi,\, \Psi) \in \mathcal{R}'_1\}$. Then, in particular, we have $\det \Psi \neq 0$, and from the precedent proof we have also $\dim \mathcal{R}_1^{\psi_0} = 10$. Let ψ_0 be a fixed point, then the coefficients ψ^i_j of Ψ satisfy a system consisting of the six homogeneous equations (B5). If, moreover, the four column vectors $\phi_k \equiv (\psi^i_k)$ $(k = 1, \ldots, 4)$ are orthogonal each to the others, then, additionally, Ψ verifies a system consisting of six multivariate quadratic equations $Q_i(\Psi) = 0$ $(i = 1, \ldots, 6)$ (the six scalar products of the four column vectors ϕ_k). Hence, let r_{ψ_0} be the smooth map such that $r_{\psi_0} : \Psi \in \mathcal{R}_1^{\psi_0} \longrightarrow (Q_1(\Psi), \ldots, Q_6(\Psi)) \in \mathbb{R}^6$, then, we can show that $\ker r_{\psi_0}$ is a nonempty four dimensional manifold (see Lemma 1, p. 11 of Ref. [Mil97]). Indeed, the tangent map Tr_{ψ_0} is regular in $\mathcal{R}_1^{\psi_0}$ because the coefficients of Tr_{ψ_0} are linear with respect to Ψ, and then, if $\det Tr_{\psi_0} = 0$, we would have the four vectors ϕ_k not linearly independent, which is not possible from the relation $\det \Psi \neq 0$. In addition, because the twelve polynomials Q_i and $\mathcal{F}_j$ are homogeneous, then the four vectors ϕ_k can be normalized, and thus, $\Psi \in O(4,\mathbb{R})$. It follows that 1) $S^{\psi_0} \equiv O(4,\mathbb{R}) \cap \mathcal{R}_1^{\psi_0}$ is not empty, and 2) S^{ψ_0} is a real semialgebraic set consisting of sixteen homogeneous multivariate polynomial equations of even degrees and one inequation. Consequently, because there are as many algebraic equations than unknowns, we obtain a nonempty *finite* set $s(\psi_0)$ of real roots $\Psi \in O(4,\mathbb{R})$ which are solutions of the system (B5) (see Ref. [Cos00] and also Chap. 2.3 of Ref. [BCR98]). Moreover, from the 'continuity of roots' property, the continuity of g on $\mathcal{M}$ and the connexity of $\mathcal{M}^2$, we deduce that the lower semi-continuous function $|s(\psi_0)|$ is constant over $\mathcal{M}^2$; And then, the set $\cup_{\psi_0\in\mathcal{M}^2} S^{\psi_0}$ is a covering of $\mathcal{M}^2$ which is universal because $\mathcal{M}^2$ is simply connected. Therefore, there is only one preimage of ψ_0 under $\alpha\times\beta$ in S^{ψ_0}. $\square$

4 Conclusion

In a four-dimensional spacetime $\mathcal{M}$ and from Theorems B.1 and B.2, we deduce that any given RPS can be *univocally* identified with a unique set of four particular emission coordinates such that the metric field $\tilde{g}$ on $\mathcal{M}$ is *ℓ-generic* in this system of coordinates. As a consequence, only four functions ν_i are needed to feature completely the (pseudo-)Riemannian geometry of the spacetime manifold once a RPS is given. In other words, a part of the geometrical structure of the spacetime manifold is transferred somehow on the RPS features. This means, in turn, that a RPS can be considered as a genuine constitutive part of a spacetime manifold and, moreover, that RPSs cannot be viewed only as physical and geometrical processes providing merely spacetime charts and atlases.

Appendix C: The "loop degeneracy"

On any spacetime manifold $\mathcal{M}$, any given metric field g with a Lorentz signature can be faithfully represented by symmetric 2-covectors in coframes with different causal types. Besides, relativistic positioning systems primarily and fundamentally favor real relativistic physical coframes comprised of exterior differentials of the so-called 'emission coordinates.' Their g-dual frames of $\{\ell\ell\ell\ell\}$ causal type are provided by bundles of light-like basis vectors tangent to the geodesic light-like beams emitted by the satellites of the positioning constellations (ℓ for light). Also, representing g in Newtonian coframes, we consider $\{ssst\}$ causal coframes (s for space and t for time). Then, we show that any metric g represented in a given $\{ssst\}$-coframe is in an algebraic topological correspondence, sometimes unexpectedly, with an infinite set—a loop in $\mathbb{R}^4$—of 2-covectors representing g in different $\{\ell\ell\ell\ell\}$-coframes. As a result, within the context of the pseudo-Riemannian geometry of the general relativity, this non-univocal correspondence radically calls into question the plain validity, in any circumstance, of the use of any $\{ssst\}$-coframe as well as the validity either of some particular irreducible parts of any computation performed with or else of any comparative interpretation of causal processes deduced from the corresponding Newtonian systems of coordinates.

1 Introduction

In linear algebra, Jordan canonical forms are, actually, ascribed to a particular type of generic "normality" for square matrices, and Sylvester's law of inertia, for instance, attributes a strong genericity to the peculiar diagonal matrices. In relativity, the diagonalizations of the matrices representing faithfully the Lorentzian metrics exhibit necessarily, as a result, specific coframes of $\{ssst\}$ causal type used in various tensor representations; which means, somehow, that Sylvester's law cannot be a genuinely relativistic property because it introduces implicitly a "classical" (nonrelativisitic) separation between space and time or, equivalently, that relativity is not generically "normal."

Besides, Relativistic Positioning Systems (RPS) aggregate (or are at the origin of) the interaction processes between, on the one hand, data transportations or signaling and, on the other hand, the relativistic physics of the data carriers (users, emitters, transmitters, etc…). The main "aggregator" is the light with its material transmitters, and relativity and the signaling of events are at the heart of these relativistic physical first systems. Subsequently, the latter produce spacetime charts for navigation with $\{\ell\ell\ell\ell\}$ causal tensor representations and for which the nondiagonalization and the vanishing of the four diagonal coefficients of any Lorentzian metric is actually the new generic "norm" or standard.

As quoted by Coll, Derrick [Der81] was probably the first to broach such considerations. Based on this work, improvements in real relativistic positioning systems were developed by authors, including Coll, Ferrando, Morales-Lladosa, Pozo [CT03, CP06, CFM06b, CFM06a, CFM09b, CFM09a, CFM10b], Bahder [Bah01, Bah03, Bah04], Rovelli [Rov02a, Rov02b], Tarantola [TKPC09, CT03], Tartaglia, Capolongo and Ruggiero [Tar10, TRC11a, RCT11, TRC11b]. It is important to note that positioning systems such as GPS, GLONASS, *Galileo* or *Beidou* are not really relativistic because they require explicit relativistic corrections in computing the spacetime positions from the four time stamps received by each user (see Ashby [Ash03] for instance). Relativity is neither truly nor inherently included in their "designs" in contrast with one of the first genuinely relativistic positioning systems, called SYPOR, designed by Coll and Tarantola [Col01b, Col02], and also with the 'Galactic Positioning System' [CT03] recently evaluated, somehow, by Tartaglia *et al.* [Tar10, TRC11a, RCT11, TRC11b], from observations of pulsars and time recordings of astrophysical data.

In the present paper, we begin with concerns similar to those of Derrick but we express differently the most general Lorentzian metrics. As a result, we deduce different consequences for the relative causal aspects of the $\{ssst\}$ and $\{\ell\ell\ell\ell\}$ coframes. More precisely, we give algebraic arguments to claim that $\{\ell\ell\ell\ell\}$-coframes are prior to $\{ssst\}$-coframes because the former can fully define the latter whereas the converse is not always possible.

In Section C 2, we review the normal forms in $\{ssst\}$ and $\{\ell\ell\ell\ell\}$ coframes and we give some consequences for the coefficients of the matrix representations of the metrics. These consequences are expressed by the algebraic equations given in Section C 3 from which a classification of their solutions is given in Section C 4. Then, in Section C 5, we provide some examples and we focus on "loop" solutions. Based on general symmetry considerations, a theorem and a conjecture are presented before the conclusion. Lastly, in two appendices, we first show why the Sturm sequences cannot be applied in the present algebraic context to classify the solutions, and second, in order to skirt this difficulty, we present and prove some general properties of cubic and quartic polynomial equations. This appendix could have been presented in a separate paper, all the more so as we give results on the signs of the roots of quartic polynomial equations not encountered, amazingly, in the literature on this, in principle, well known subject. But, we would have been faced with objections since the existing efficient method of Sturm sequences with its four fundamental invariants (the Cayley's 'catalecticant' I, *i.e.*, the discriminant $\triangle$, Σ, etc… [Coh93, Cos00, Cre01]) is already available and sufficient. Then, to persuade of the need for the alternative method and the results presented in this last appendix, we ought to give examples for which the Sturm method cannot be applied contrary to our results. But, precisely the present paper gives such an example which is, moreover, linked to the fundamental topological relations between two causal tensor representations of the metric fields and their uses, in particular, in RPS.

2 Generic normal forms of $\{ssst\}$ or $\{\ell\ell\ell\ell\}$ causal types for metrics

a The generic normal forms $G_\perp$ and G_ϵ

A spacetime metric g is a symmetric 2-form (*i.e.*, 2-covector) of Lorentzian type defined on a spacetime manifold $\mathcal{M}$ of dimension $n = 4$. Let $U \subset \mathcal{M}$ be an open neighborhood of a given event $e \in \mathcal{M}$. Then, a local coordinate chart $u \equiv (u_1, \ldots, u_4) : U \longrightarrow \mathbb{R}^4$ is said to be *pseudo-orthogonal* or Lorentzian on U if the spacetime metric g is Lorentzian and can be represented in the form

$$g \equiv \sum_{i=1}^{n=4} a_{ii}(u)\, du_i \otimes du_i\,. \tag{C1}$$

Then, we also have *pseudo-orthogonal* coordinates u_i. The 1-forms $\alpha_i(u)$ $(i = 1, \ldots, 4)$ such that

$$\alpha_i \equiv \sqrt{|a_{ii}(u)|}\, du_i \tag{C2}$$

form a *pseudo-orthonormal* or Lorentzian coframe $\mathcal{B} \equiv \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then, on the pseudo-Riemannian manifold $\mathcal{M}$ and with a suitable choice of indices, g can be put in the form

$$g \equiv \pm \left(\alpha_4 \otimes \alpha_4 - \sum_{k=1}^{3} \alpha_k \otimes \alpha_k \right). \tag{C3}$$

Moreover, we have[45]

$$\alpha_i \wedge d\alpha_i = 0\,. \tag{C4}$$

Then, from the Frobenius theorem, each 1-form α_i is a multiple of an exact 1-form [Rov02a, see, for instance, formula (22)].[46] Nevertheless, the relations (C4) are not always satisfied when the

[45] We shall throughout this appendix adhere to the convention that latin indices shall take on values from 1 to 4 only.

[46] Indeed, from (C4), we deduce that there exist four 1-forms σ_i such that $d\alpha_i = \sigma_i \wedge \alpha_i$. But also, because we have four independent 1-forms α_i in a four dimensional manifold, there then exists a matrix M and a local chart with local coordinates $x \equiv \{x_1, \ldots, x_4\}$ such that $\alpha_i = \sum_{j=1}^{4} M_i^j(x)\, dx_j$. Hence, we obtain $d\alpha_i = \sum_{j=1}^{4} (M^{-1}\, dM)_i^j \wedge \alpha_j$. Then, from the relations $(M^{-1}\, dM)_i^j = \delta_i^j\, \sigma_i$, we deduce easily that for all of the indices i and j the σ_is are the exact forms $\sigma_i \equiv d\ln(M_i^j) \equiv du_i$; and then, the result because $d(e^{-u_i}\alpha_i) \equiv 0$.

metric g is in the form (C3), *i.e.*, when it is decomposed in an $\{ssst\}$-coframe $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. In other words, pseudo-orthogonal (Lorentzian) $\{ssst\}$-coframes $\mathcal{B}$ do not always exist for any spacetime manifold $\mathcal{M}$. Actually, we can show that the relations (C4) are satisfied if and only if the covariant components $R_{ij,kl}$ of the Riemann tensor defined by g vanish whenever all i, j, k, and l are distinct [BCG+91, pp.287–292][Bry99, §7].[47] We call these conditions on the Riemann tensor the BC3G conditions. Besides, it can be shown that they are equivalent to the vanishing of all of the corresponding covariant components of the Weyl curvature tensor. Obviously, it follows that the BC3G conditions are necessarily satisfied when $\mathcal{M}$ is conformally flat.

In general relativity, the BC3G conditions are always implicitly verified because local pseudo-orthogonal $\{ssst\}$ coordinate charts refer explicitly to the unicity of the $\{ssst\}$ causal class of coframes defined on $\mathcal{M}$. Indeed, if the BC3G conditions are satisfied, then, starting with a representation $\sum_{i,j=1}^n b_{ij}\, dx_i \otimes dx_j$ of g, there always exists a local diffeomorphism $f_k(u_i) \equiv x_k$ such that g is represented in the diagonal form (C1) with respect to the pseudo-orthogonal coordinates u_k. In other words, if the BC3G conditions are satisfied there exists only one $\{ssst\}$ causal class univocally associated with g. This unique causal class then defines a unique set of diffeomorphic pseudo-orthogonal $\{ssst\}$-coframes $\mathcal{B}$. On the other hand, if the BC3G conditions are not satisfied, it follows that the formulas (C1) and (C4) might not be satisfied both together. Consequently, we might have, for instance, two $\{ssst\}$-coframes, namely, $\mathcal{B} \equiv \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $\mathcal{B}' \equiv \{\rho_1, \rho_2, \rho_3, \rho_4\}$, such that $g \equiv \alpha_4 \otimes \alpha_4 - \sum_{k=1}^3 \alpha_k \otimes \alpha_k$ and $g \equiv \rho_4 \otimes \rho_4 - \sum_{k=1}^3 \rho_k \otimes \rho_k$, but with $\alpha_i \wedge d\alpha_i = 0$ and $\rho_j \wedge d\rho_j \neq 0$. Then, $\mathcal{B}$ and $\mathcal{B}'$ might be not diffeomorphic if the BC3G conditions are not satisifed, and therefore, we obtain two non-equivalent $\{ssst\}$ causal classes associated with g (or $\mathcal{M}$), each containing $\mathcal{B}$ or $\mathcal{B}'$.

Furthermore, considering the general case with the conditions BC3G not necessarily satisfied again, then, in each $\{\ell\ell\ldots\ell\}$-coframe we have the following result. Let $\tilde{\tau}$ be a given local emission coordinate chart of a given $\{\ell\ell\ldots\ell\}$ causal class with coordinates $\tilde{\tau}_j$ such that $\tilde{\tau} \equiv (\tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_n) : U \subset \mathcal{M} \longrightarrow \mathbb{R}^n$ on an n-dimensional spacetime $\mathcal{M}$, then the metric g must

[47] In fact, the proof is given for orthogonal coordinate charts and Euclidean metrics on $\mathbb{R}$. However, because the proof is analytical, the latter remains completely valid on the field $\mathbb{C}$ of complex numbers considering an Euclidean metric $g = \sum_{j=1}^4 \alpha_j \otimes \alpha_j$ on $\mathbb{C}$ with $\alpha_k \equiv i\sqrt{|a_{kk}|}\, du_k$ for three of the four indices j and, additionally, keeping the same dimensions of the various varieties or manifolds but on $\mathbb{C}$. See also, for instance, the conclusion P.15 § 4.2 "The Lorentzian case" in [GV09].

satisfy

$$g \equiv \sum_{i<j=1}^{n} e_{ij}\, d\tilde{\tau}_i \odot d\tilde{\tau}_j\,, \tag{C5}$$

where, for all i, $j = 1, \ldots, n$, we have $\alpha \odot \beta \equiv (\alpha \otimes \beta + \beta \otimes \alpha)/2$, $e_{ij} = e_{ji} \neq 0$ and $e_{ii} = 0$. However, it can be shown that if $n > 4$, we cannot always find new emission coordinates τ_k and n 1-forms λ_k such that

$$g \equiv 2\epsilon \sum_{i<j=1}^{n} \lambda_i \odot \lambda_j\,, \tag{C6}$$

where $\epsilon = \pm 1$ and $\lambda_i \equiv \nu_i(\tau)\, d\tau_i$ where the functions $\nu_i(\tau)$ are positive. Nevertheless, if $n \leqslant 4$, these new emission coordinates τ_i always exist (see Appendix B).

Hence, we must specify the relations between the representations (C6) and the diagonal representations (C3) of g at any given event $e \in \mathcal{M}$. As the aim of this paper, we show that given an $\{ssst\}$-coframe via a diagonal representation (C3) of g, we can always have an uncountable set of corresponding $\{\ell\ell\ell\ell\}$-coframes for g represented by (C6), whereas, on the contrary, given an $\{\ell\ell\ell\ell\}$-coframe, we necessarily have only a finite set of $\{ssst\}$-coframes for g. Therefore, this result shows in turn that $\{\ell\ell\ell\ell\}$-coframes are prior to $\{ssst\}$-coframes since changes from $\{ssst\}$ to $\{\ell\ell\ell\ell\}$ coframes are more indeterminate than the reverse direction. Moreover, to certain $\{ssst\}$-coframes associated with g, there correspond a set of $\{\ell\ell\ell\ell\}$-coframes with the topology of S^1.

The metric g has symmetric matrix representations and, in a particular, g can be represented in a given dual $\{ssst\}$-frame by a diagonal matrix $G_\perp$ with the terms β_k as diagonal coefficients. In another given dual $\{\ell\ell\ell\ell\}$-frame, g can be represented by a matrix G_ϵ such that (see Appendix B).:

$$G_\epsilon \equiv \epsilon \begin{pmatrix} 0 & \nu_1\,\nu_2 & \nu_1\,\nu_3 & \nu_1\,\nu_4 \\ \nu_2\,\nu_1 & 0 & \nu_2\,\nu_3 & \nu_2\,\nu_4 \\ \nu_3\,\nu_1 & \nu_3\,\nu_2 & 0 & \nu_3\,\nu_4 \\ \nu_4\,\nu_1 & \nu_4\,\nu_2 & \nu_4\,\nu_3 & 0 \end{pmatrix}, \tag{C7}$$

where the products $\nu_i\,\nu_j$ are necessary nonvanishing whenever $i \neq j$. Then, the relation between these two representations of g can be expressed as follows. Let h be a local diffeomorphism such that $h(u) \equiv \tau$ where u and τ are the coordinates of the same event e in the two different systems of coordinates and J the invertible Jacobian matrix of h. Then, J satisfies the relation $G_\perp = {}^tJ\,G_\epsilon\,J$ where ${}^tJ$ denotes the transpose matrix of J because g is a (0, 2) tensor rather than a mixed (1, 1) tensor. However, according to a classical theorem of linear algebra [CM74, see Corollary 1 and Proposition p. 50], if M is a finite symmetric square matrix, then there always exists an orthogonal matrix P such that $N \equiv {}^tP\,M\,P = P^{-1}\,M\,P$ is diagonal. Hence, in particular, we can choose $G_\perp$ such that J is an element of the orthogonal group $O(4)$, *i.e.*, ${}^tJ = J^{-1}$, and then the characteristic polynomials of $G_\perp$ and G_ϵ must be equal (*i.e.*, invariant): $\det(g - \lambda\,id) = \det(G_\perp - \lambda\mathcal{I}) = \det(G_\epsilon - \lambda\mathcal{I})$, with $\det(g - \lambda\,id) = \lambda^4 - s_1\,\lambda^3 + s_2\,\lambda^2 - s_3\,\lambda + s_4$.

It can be shown that

$$s_1 = \sum_{i=1}^{4} \beta_i = 0\,, \tag{C8a}$$

$$s_2 = \sum_{i<j=1}^{4} \beta_i\beta_j = -\sum_{i<j=1}^{4} (\nu_i\nu_j)^2\,, \tag{C8b}$$

$$s_3 = \sum_{i<j<k=1}^{4} \beta_i\beta_j\beta_k = 2\,\epsilon \sum_{i<j<k=1}^{4} (\nu_i\nu_j\nu_k)^2\,, \tag{C8c}$$

$$s_4 = \prod_{i=1}^{4} \beta_i = -3\prod_{i=1}^{4}(\nu_i)^2\,. \tag{C8d}$$

Hence, if the values of the functions ν_k at e are given, then, necessarily, the number of values of the functions β_i at e is finite because univariate polynomials (of degree 4) have only a finite number of roots. But, conversely, if the values of the functions β_i are given we have no certainty that the number of values for the functions ν_k at e is finite because we have only three algebraic equations for the ν_ks not related to a fully defined particular univariate polynomial (and in particular of degree 4). Moreover, because the relations (C8) are deduced from characteristic polynomials, they are independent on the $\{ssst\}$ and $\{\ell\ell\ell\ell\}$ coframes—though the values of the βs and the νs are separately dependent on the systems of coordinates—and then, they are, somehow, the algebraic representations of the topological relation between the $\{ssst\}$ and $\{\ell\ell\ell\ell\}$ coframes.

Besides, we see that $G_\perp$ must be traceless ($s_1 = 0$). Then, we must assume that any similar matrix $G'_\perp$ that represents g and that is diagonal in an $\{ssst\}$-coframe can always be put in an associated traceless diagonal form $G_\perp$ in correspondence with a G_ϵ matrix. This is a very strong assumption or constraint. To obtain the matrix $G_\perp$ from the matrix $G'_\perp$, we must separate the scaling on the α'_k's ($k \neq 4$) and α'_4, *i.e.*, the scaling factors applied to α'_4, for instance, must only depend on the "time" parameterization of the coordinate curve parameterized by u'_4 and defined by α'_4. Thus, we can make the following proposal:

> *If the space and time coordinates make "sense", and if RPS give "complete" first descriptions of the spacetime manifold $\mathcal{M}$ from the spacetime charts they yield, then $\mathcal{M}$ must be a trivial 'bi-conformal' foliation; i.e., $\mathcal{M} \simeq \mathcal{S}\times\mathcal{T}$, where g is defined separately up to two conformal factors: one on the space manifold $\mathcal{S}$ and the other on the time manifold $\mathcal{T}$.*

The meaning of the words "make sense" in this proposition will be clarified in the conclusion based on the results presented below.

b The coefficients of $G_\perp$ and G_ϵ

From (C8), we can deduce a list of properties and remarks related to the coefficients of $G_\perp$ and G_ϵ:

A– From (C8d), we deduce that $s_4 = \det g < 0$. Then, necessarily g has one of the two Lorentzian signatures $\sigma \equiv (3\pm, \mp)$ (or also $\sigma \equiv \pm 2$) because there must be an odd number of positive (or negative) coefficients β_i. We assume that the β_k ($k = 1, 2, 3$) have the same sign and β_4 has the opposite sign. Moreover, $\nu_i \neq 0$ for all $i = 1, \ldots, 4$.

B– We have $s_1 = 0$ from (C8a). Thus, s_3 can be rewritten depending only on the coefficients β_k ($k = 1, 2, 3$): $s_3 = -(\beta_1 + \beta_2)(\beta_1 + \beta_3)(\beta_2 + \beta_3)$. As a result, we must have $\mathrm{sgn}(s_3) = \mathrm{sgn}(\beta_4) = \epsilon$, and then $\sigma = (-3\epsilon, \epsilon)$ (or $\sigma \equiv -2\epsilon$).

C– The chronological order involves the following [CP06, see Properties 1 and 2]:

1. If k_1 and k_2 are two future-directed light-like vectors of an $\{\ell\ell\ell\ell\}$ frame such that $d\tau_i(k_j) = \delta_{ij}$ for $i = 1, \ldots, 4$ and $j = 1, 2$, then the $d\tau_k$ ($k = 1, 2$) are causally

future-directed null 1-forms (covectors); *i.e.*, "$d\tau_k > 0$," which means that the values of τ_k increase when moving towards the future. However, the coefficient $G_\epsilon(k_1, k_2) = \epsilon\,\nu_1\nu_2$ must also have the same sign as ϵ; thus, $\nu_1\nu_2 > 0$, and more generally, $\nu_i\nu_j > 0$ ($i \neq j = 1, \ldots, 4$). Hence, we deduce that the four nonvanishing functions ν_k have the same sign.

2. If the 1-forms λ_k are all exact, then, in particular, we can define the four coordinates τ_h in such a way that $\lambda_i \equiv d\tau_i$. Therefore, we deduce more generally that the 1-forms λ_k must always be future-directed null covectors similar to the $d\tau_k$s. Consequently, we have

$$\nu_i > 0\,, \qquad i = 1, \ldots, 4. \tag{C9}$$

Henceforth, throughout the paper, we set $\epsilon = -1$, and thus $s_i < 0$ for $i = 2, 3, 4$.

3 The systems of algebraic equations

a The algebraic system in dimension 4

Because the functions ν_i are all nonvanishing functions, we can make the following change of coordinates ($i = 1, \ldots, 4$):

$$x_i \equiv \frac{1}{(\nu_i)^2} > 0\,. \tag{C10}$$

Then, dividing by $\prod_{i=1}^4 (\nu_i)^2$ the three last relations of (C8), we obtain the following equivalent relations:

$$h_1 \equiv \sum_{i=1}^4 x_i = \frac{3s_3}{2s_4} > 0\,, \tag{C11a}$$

$$h_2 \equiv \sum_{i<j=1}^4 x_i x_j = \frac{3s_2}{s_4} > 0\,, \tag{C11b}$$

$$h_4 \equiv \prod_{i=1}^4 x_i = -\frac{3}{s_4} > 0\,, \tag{C11c}$$

and we define h_3 such that

$$h_3 \equiv \sum_{i<j<k=1}^{4} x_i x_j x_k > 0\,. \tag{C11d}$$

The system of algebraic equations (C11) is similar to the (C8), but, in contrast to the four s_i, h_3 is free to vary and thus it is not considered hereafter. Additionally, the variables x_i are the roots of the following polynomials varying with h_3:

$$H(z) = z^4 - h_1\,z^3 + h_2\,z^2 - h_3\,z + h_4\,. \tag{C12}$$

b The reduced algebraic system in dimension 3

To solve the algebraic system (C11), we again change the variables. We make the following assumptions which are similar to those of Ferrari [Mut80]:

$$x_1 = \frac{1}{2}\left(\frac{1}{2}h_1 - \sqrt{\alpha} - \sqrt{\beta} - \sqrt{\gamma}\right), \tag{C13a}$$

$$x_2 = \frac{1}{2}\left(\frac{1}{2}h_1 - \sqrt{\alpha} + \sqrt{\beta} + \sqrt{\gamma}\right), \tag{C13b}$$

$$x_3 = \frac{1}{2}\left(\frac{1}{2}h_1 + \sqrt{\alpha} + \sqrt{\beta} - \sqrt{\gamma}\right), \tag{C13c}$$

$$x_4 = \frac{1}{2}\left(\frac{1}{2}h_1 + \sqrt{\alpha} - \sqrt{\beta} + \sqrt{\gamma}\right). \tag{C13d}$$

Then, (C11a) is always satisfied regardless of the positive or vanishing real values for α, β and γ, which are the solutions of a system of two algebraic equations deduced from (C11b) and (C11c):

$$\alpha + \beta + \gamma = \frac{3}{4}{h_1}^2 - 2h_2 \geqslant 0\,, \tag{C14a}$$

$$h_1\sqrt{\alpha\beta\gamma} + \alpha\beta + \beta\gamma + \alpha\gamma = \frac{1}{16}\left({h_1}^2 - 4h_2\right)^2 - 4h_4 \geqslant 0\,. \tag{C14b}$$

Introducing two auxiliary real variables u and v, we can rewrite (C14) as a system of four algebraic equations:

$$\alpha + \beta + \gamma = \frac{3}{4}{h_1}^2 - 2h_2 \equiv r_1 \geqslant 0\,, \tag{C15a}$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = h_1\,v^2 \equiv r_2 \geqslant 0\,, \tag{C15b}$$

$$\alpha\beta\gamma = u^4 \equiv r_3 \geqslant 0\,, \tag{C15c}$$

and

$$u^2 + v^2 = \frac{1}{h_1}\left(\frac{1}{16}\left({h_1}^2 - 4h_2\right)^2 - 4h_4\right) \geqslant 0\,. \tag{C15d}$$

Based on these expressions and variables, α, β and γ can be equivalently considered as the nonnegative roots of the polynomial $P(z)$ of degree 3 such that

$$P(z) \equiv z^3 - r_1 z^2 + r_2 z - r_3\,. \tag{C16}$$

Thus, from (C15a) and (C15d), we obtain the first set of necessary conditions to have such positive solutions for the (C11):

$$h_5 \equiv \frac{1}{h_1}\left(\frac{1}{16}\left({h_1}^2 - 4h_2\right)^2 - 4h_4\right) \geqslant 0\,, \tag{C17a}$$

$$h_6 = r_1 \equiv \frac{3}{4}{h_1}^2 - 2h_2 \geqslant 0\,. \tag{C17b}$$

Or, equivalently, with $h_i > 0$ ($i = 1, \ldots, 4$), we can also write

$$0 < h_4 \leqslant \frac{1}{64}\left({h_1}^2 - 4h_2\right)^2, \tag{C18a}$$

$$0 < h_2 \leqslant \frac{3}{8}{h_1}^2\,. \tag{C18b}$$

In what follows, the unfolding of the system (C15) will reveal a kind of "loop-degeneracy" for the correspondence between $G_\perp$ and G_ϵ and thus between $\{ssst\}$ and $\{\ell\ell\ell\ell\}$ frames as indicated in the introduction.

4 The domains of algebraic resolution

As shown in Section C 3 b, the roots of the cubic polynomial $P(z)$ defined by (C16) must be three (counting multiplicities) real nonnegative roots. Thus, we must compute the discriminant $\triangle_P$ of $P(z)$ such that

$$\triangle_P \equiv 18\, r_1\, r_2\, r_3 + {r_1}^2\, {r_2}^2 - 4\, {r_1}^3\, r_3 - 4\, {r_2}^3 - 27\, {r_3}^2\,. \tag{C19}$$

We assume that the polynomial $\triangle_P$ depends on the variable r_2 which is not fixed in our description. In particular, we have

$$r_3 = \frac{1}{{h_1}^2}\left(r_2 - h_1 h_5\right)^2. \tag{C20}$$

Moreover, from (C15b) and (C15d), we have $0 \leqslant r_2 \leqslant h_1\, h_5$. We then obtain from the expression $P(z) \equiv z^3 - h_6\, z^2 + r_2\, z - (r_2 - h_1 h_5)^2/{h_1}^2$ of $P(z)$ the following discriminant:

$$\triangle_P(r_2) \equiv \frac{1}{{h_1}^4} \sum_{i=0}^{4} a_i\, {r_2}^i\,, \tag{C21}$$

where

$$a_4 = -27 < 0\,, \tag{C22a}$$

$$a_3 = 2\, h_1\, (9\, h_6\, h_1 + 54\, h_5 - 2\, {h_1}^3)\,, \tag{C22b}$$

$$a_2 = h_1^2\, ({h_1}^2\, {h_6}^2 - 162\, {h_5}^2 - 36\, h_1\, h_6\, h_5 - 4\, {h_6}^3)\,, \tag{C22c}$$

$$a_1 = 2\, {h_1}^3\, h_5\, (54\, {h_5}^2 + 9\, h_5\, h_1\, h_6 + 4\, {h_6}^3) \geqslant 0\,, \tag{C22d}$$

$$a_0 = -{h_1}^4\, {h_5}^2\, (4\, {h_6}^3 + 27\, {h_5}^2) \leqslant 0\,. \tag{C22e}$$

We want to know the interval for r_2 where $\triangle_P(r_2) \geqslant 0$. Indeed, if $\triangle_P(r_2) \geqslant 0$, then $P(z)$ has only real roots (see the Appendix E 1). We must know also the signs of these real roots.

Remark 17. *We could have used Sturm's theorem because we want to know how many roots of $\triangle_P(r_2)$ are between 0 and $h_1 h_5$. However, we should have discussed the signs of tremendous polynomials of degree at least 4 in h_1, h_5 or h_6 again (see the Appendix D). The remark would be the same using the Sylvester–Hermite method.*

Remark 18. *We seek some conditions on the variables h_1, h_5 and h_6 so that $\triangle_P(r_2)$ remains positive for r_2 in an open connected subset in $\mathbb{R}$. Indeed, our goal is to find positive roots α, β and γ varying continuously to exhibit what we call "loop-degeneracy"; i.e., embeddings of sets homeomorphic to S^1 in $\mathbb{R}^4$. Hence, we will focus the discussion below on addressing which of the different situations encountered satisfies this condition of continuity.*

a The roots if $r_2 = 0$

Then, we have $r_3 = {h_5}^2$, $P(z) = z^3 - h_6\, z^2 - {h_5}^2$ and $\triangle_P(0) = -{h_5}^2(4\, {h_6}^3 + 27\, h_5) \leqslant 0$. If $\triangle_P(0) < 0$ and $r_3 \neq 0$, we deduce from Theorem E.4 (Appendix E 3) that there exists only one simple root, and thus, two other complex conjugate simple roots. Therefore, we do not have three real roots. There remains the case $\triangle_P(0) = 0$, and from Theorem E.4 again, if $r_3 \neq 0$, we must calculate the quantity $L \equiv 27\,r_3 - {r_1}^3 = 27\,{h_5}^2 - {h_6}^3$. In this case, we have a triple positive root whenever $L = 0$ or a simple positive root and a negative double root otherwise because $r_2 = 0$. Hence, in order to have three positive roots α, β and γ, we must have $L = 0$, *i.e.*, $27\,{h_5}^2 = {h_6}^3$. As a result, we obtain $\triangle_P(0) = -135\,{h_5}^4$ and $\triangle_P(0) = 0$. It follows that $h_5 = h_6 = 0$ and $P(z) \equiv z^3$. But then $r_3 = 0$ and we obtain a contradiction. Hence, we have necessarily $L \neq 0$ if $r_3 \neq 0$ and no three nonnegative real roots unless $r_3 = 0$. In this latter case, we obtain $P(z) = z^2(z - h_6)$ and we have three nonnegative real roots. Nevertheless, we obtain only a finite set of roots. Therefore, we assume in the sequel that $0 < r_2 \leqslant h_1\,h_5$ and, in particular, $h_5 > 0$.

b The roots if $r_2 = h_1 h_5$

First, we deduce that $r_3 = 0$ from (C20). Then, we have $P(z) = z(z^2 - h_6 z + h_1 h_5)$ and $\triangle_P(h_1\,h_5) = h_1^2\,h_5^2\,(h_6^2 - 4\,h_1\,h_5)$. The factor $\triangle_R \equiv h_6^2 - 4\,h_1\,h_5$ in $\triangle_P(h_1\,h_5)$ is the discriminant of the polynomial factor $R(z) \equiv z^2 - h_6\,z + h_1 h_5$ in $P(z)$. Therefore, if we assume that

$$h_7 \equiv h_6^2 - 4\,h_1\,h_5 \geqslant 0\,, \tag{C23}$$

then the variables α, β and γ are equal to the three nonnegative values:

$$z_0 \equiv 0, \qquad z_+ \equiv \frac{1}{2}\left(h_6 + \sqrt{h_6^2 - 4\,h_1\,h_5}\right), \qquad z_- \equiv \frac{1}{2}\left(h_6 - \sqrt{h_6^2 - 4\,h_1\,h_5}\right). \tag{C24}$$

Hence, to seek for an infinite set of roots, we assume from now on and throughout that $r_3 \neq 0$ and $r_3 \neq {h_5}^2$ or, equivalently, $0 < r_2 < h_1 h_5$.

c The roots if $\triangle_P(r_2) \geqslant 0$ with $r_2 \in]0,\, h_1 h_5[$

Necessarily, we have $r_3 \in]0, {h_5}^2[$. If $\triangle_P(r_2) = 0$, we have to compute the parameter $L \equiv 27 r_3 - r_1^3$ (Appendix E 3, Theorem E.4). Hence, if $L = 0$ the roots α, β and γ are equal and positive (the sign of r_3). Furthermore, if $L \neq 0$, we have a simple positive (the sign of r_3) root, and a double positive root if and only if $r_2 > 0$ and $r_1\,r_3 > 0$, *i.e.*, $r_1 > 0$. Therefore, we always have positive roots α, β and γ, two of them being equal if and only if $r_1 > 0$ when $L \neq 0$. Moreover, these three positive roots become continuous functions depending on r_2.

If $\triangle_P(r_2) > 0$, again from Theorem E.4, we have three distinct simple real positive roots α, β and γ if and only if $r_i > 0$ for all $i \in \{1, 2, 3\}$. Therefore, we obtain the same condition as above when $\triangle_P(r_2) = 0$, and again, these roots depend continuously on r_2.

d The nonnegative domain of $\triangle_P(r_2)$ with $r_2 \in]0, h_1 h_5[$

We want to know the subset in the interval $I \equiv [0, h_1 h_5]$ in which $\triangle_P(r_2) \geqslant 0$. Thus, we must compute the discriminant $\triangle_4(h_5)$ of the polynomial $\triangle_P(r_2)$ of degree 4 (see Appendix E 2). Indeed, we must know what are the types of the roots of $\triangle_P(r_2)$, whether complex or real, simple or not. However, we focus on the conditions for the existence of simple positive (real) roots of $\triangle_P(r_2)$ to ensure that there are changes of signs at these values of r_2. Moreover, the 'standard' polynomial $\widetilde{P}(z)$ defined by translation from $P(z)$ is the polynomial such that

$$\widetilde{P}(z) \equiv P\left(z + \frac{h_6}{3}\right) = z^3 + p(r_2)\, z + q(r_2)\,, \tag{C25}$$

where

$$p(r_2) = r_2 - \frac{{h_6}^2}{3}\,, \tag{C26a}$$

$$q(r_2) = \frac{h_6}{3}\left(r_2 - \frac{2}{9}\,{h_6}^2\right) - \frac{1}{{h_1}^2}\,(r_2 - h_1\, h_5)^2\,. \tag{C26b}$$

The discriminant $\triangle_P(r_2)$ of $P(z)$ is equal to the discriminant $\triangle_{\widetilde{P}}(r_2)$ of $\widetilde{P}(z)$, and thus it is also defined by the well known relation $\triangle_P(r_2) = \triangle_{\widetilde{P}}(r_2) = -(4\, p(r_2)^3 + 27\, q(r_2)^2)$. Then, because $\triangle_P(r_2) \geqslant 0$, we must have $p(r_2) \leqslant 0$, *i.e.*, $0 < r_2 \leqslant {h_6}^2/3$, and, in particular, $h_6 > 0$.

If $p(r_2) = 0$ and $q(r_2) \neq 0$ or if $q(r_2) = 0$ and $p(r_2) \leqslant 0$, then $\triangle_P(r_2)$ has only real roots which are easy to find.

Henceforth, as a result, we assume that

$$p(r_2) < 0, \qquad q(r_2) \neq 0, \qquad h_5 > 0, \qquad h_6 > 0. \tag{C27}$$

It follows from (C26a) that $r_2 < {h_6}^2/3$. Then, we deduce the following domain of variation $\mathcal{D}$ for r_2:

$$r_2 \in \mathcal{D} \equiv \Big]0,\, \min\Big(h_1 h_5,\, \tfrac{{h_6}^2}{3}\Big)\Big[\,. \tag{C28}$$

Moreover, using Maplesoft to implement the Cantor-Zassenhaus factorization algorithm [CZ81], we see that $\triangle_4(h_5)$ can be factored:

$$\triangle(h_5) = 2^{14}\, 3^9\, \frac{{h_5}^2}{{h_1}^3}\,(h_5 - h_{5,0})\,(h_5 - h_{5,+})^3\,(h_5 - h_{5,-})^3\,, \tag{C29}$$

where

$$h_{5,0} = \frac{1}{64 h_1}(4h_6 - {h_1}^2)^2, \qquad\qquad h_{5,\pm} = \frac{h_6}{9h_1}\left(3\,h_6 \pm h_1\,\sqrt{3h_6}\,\right). \tag{C30}$$

We notice that

$$h_5 \leqslant h_{5,0} \quad\Longrightarrow\quad h_1 h_5 < \frac{{h_6}^2}{4} < \frac{{h_6}^2}{3} \quad\Longrightarrow\quad r_2 \in]0, h_1 h_5[\,, \tag{C31a}$$

$$h_{5,+} \leqslant h_5 \quad\Longrightarrow\quad h_1 h_5 > \frac{{h_6}^2}{3} \quad\Longrightarrow\quad r_2 \in \Big]0,\, \tfrac{{h_6}^2}{3}\Big[\,, \tag{C31b}$$

and therefore, it suffices, in particular, that $h_{5,+} \leqslant h_5 \leqslant h_{5,0}$ to have $r_2 \in \mathcal{D}$. Furthermore, we define the coefficients σ_k $(k = 1, \ldots, 4)$ such that

$$\sigma_1 \equiv -\frac{a_3}{a_4}\,, \qquad \sigma_2 \equiv \frac{a_2}{a_4}\,, \qquad \sigma_3 \equiv -\frac{a_1}{a_4}\,, \qquad \sigma_4 \equiv \frac{a_0}{a_4}\,, \tag{C32}$$

and then, the polynomial Q such that

$$Q(r_2) \equiv -\frac{{h_1}^4}{27}\,\triangle_P(r_2) = {r_2}^4 - \sigma_1\,{r_2}^3 + \sigma_2\,{r_2}^2 - \sigma_3\,r_2 + \sigma_4\,. \tag{C33}$$

Because we have a proportionality between the polynomials Q and $\triangle_P$, the discriminant of Q is also $\triangle_4(h_5)$, *i.e.*, $\triangle_Q(h_5) \equiv \triangle_4(h_5)$. Now, checking the cases in Theorem E.5 for which 1) $\sigma_4 > 0$ and $\sigma_3 > 0$, and 2) there is at least one simple positive root, we find suitable only the following cases:

Case 1: $\triangle_4(h_5) < 0$. Two positive simple roots if $\mathrm{sgn}(\sigma_1 + \sigma_3) > 0$, *i.e.*, $a_1 + a_3 > 0$.

Case 2a: $\triangle_4(h_5) > 0$. Four positive simple roots if $\mathrm{sgn}(\sigma_1) \equiv \mathrm{sgn}(a_3) > 0$ and $\mathrm{sgn}(\sigma_1\sigma_2\sigma_3) \equiv \mathrm{sgn}(a_1 a_2 a_3 a_4) > 0$, or, equivalently, if $a_2 < 0$ and $a_3 > 0$.

Case 2b: $\triangle_4(h_5) > 0$. Two positive simple roots and two negative simple roots if $a_2 \geqslant 0$ or $a_3 \leqslant 0$.

Case 3: $\triangle_4(h_5) = 0$. From Theorem E.3, we must have $K_3 = H_1 = 0$ and $H_2 > 0$, and then we have two nonvanishing simple roots u_1 and u_2 and a double nonvanishing root $u_3 \equiv \sigma_1/4$. Then, from Theorem E.5, because $\sigma_4 > 0$, all of these roots u_1, u_2 and u_3 have the sign of σ_1, *i.e.*, the sign of a_3. Therefore, the simple roots u_1 or u_2 are both positive.

Lastly, there is another remarkable useful value for $\triangle_P(r_2)$:

$$\triangle_P(h_1\,h_5) = h_1^2\,h_5^2\,(h_6^2 - 4\,h_1\,h_5)\,. \tag{C34}$$

Therefore, if $h_5 \leqslant h_{5,0}$, we deduce that $h_1 h_5 < h_6^2/4$ and, as a result, $\triangle_P(h_1\,h_5) > 0$ and $r_2 \in [0, h_1 h_5]$. Moreover, because $\triangle_P(0) < 0$, then, necessarily, at least one positive root of $\triangle_P(r_2)$ is greater than $h_1\,h_5$ and another one smaller. Hence, the case for which $\triangle_P(h_1\,h_5) \leqslant 0$ is obtained whenever $h_5 > h_{5,0}$.

Moreover, we could look for criteria to determine the positions of the positive roots of $\triangle_P(r_2)$ with respect to $h_1 h_5$ or $h_6^2/3$. For we could define, for instance, the "conjugate" polynomial $\triangle_P^*(s_2) \equiv \triangle_P(s_2 + h_1 h_5)$, and we could seek the number of its non-positive roots. We could do the same with the other "conjugate" polynomial $\triangle_P^{**}(s_2) \equiv \triangle_P(s_2 + h_6^2/3)$. Again, we could discuss the signs of the polynomials of degree 4 such as the one for the case 1 above with $a_1 + a_3$. But, because our goal is mainly to exhibit an uncountable set of solutions, we give examples only and we discuss the general situation below.

5 Examples and "loop" solutions

We give three fundamental examples differing from the sign of $\triangle_P(h_1 h_5)$:

1. FIG. C.1: $h_1 = 3$, $h_2 = 3.05$, $h_4 = 0.1$. We obtain $h_5 = 0.08$, $h_6 = 0.65$, $h_6^2/3 \simeq 0.14\cdots < h_1 h_5 = 0.24$, $\triangle(h_5) > 0$, $\triangle_P(h_1 h_5) < 0$, σ_1 negative, and the σ_js positive ($j = 2, 3, 4$).

2. FIG. C.2: $h_1 = 4$, $h_2 = 5.5$, $h_4 = 0.5$. We obtain $h_5 = 0.0625$, $h_6 = 1$, $h_6^2/3 \simeq 0.33\cdots > h_1 h_5 = 0.25$, $\triangle(h_5) > 0$, $\triangle_P(h_1 h_5) = 0$, σ_1 and σ_2 negative, σ_3 and σ_4 positive.

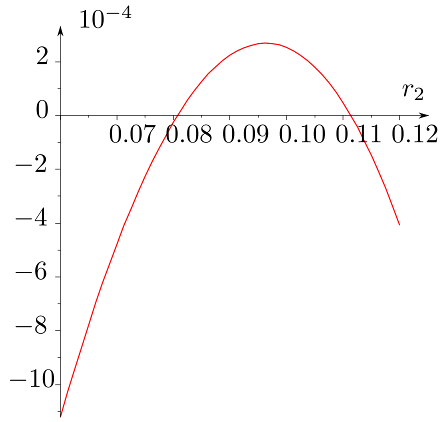


Figure C.1. $\triangle_P(r_2)$ with $\triangle_P(h_1 h_5) < 0$.

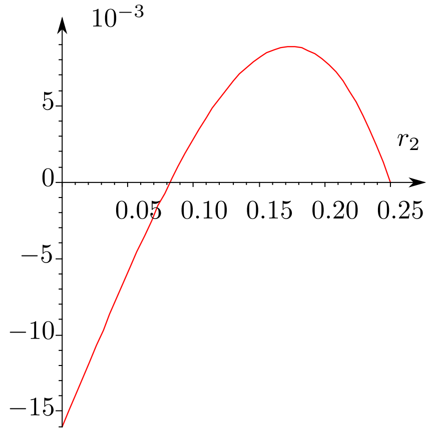


Figure C.2. $\triangle_P(r_2)$ with $\triangle_P(h_1 h_5) = 0$.

3. FIG. C.3: $h_1 = 3$, $h_2 = 1$, $h_4 = 0.3$. We obtain $h_5 \simeq 0.1208\ldots$, $h_6 = 4.75$, $h_6^2/3 \simeq 7.52\cdots > h_1 h_5 = 0.362\ldots$, $\triangle(h_5) > 0$, $\triangle_P(h_1 h_5) > 0$, and all of the σ_ks are positive.

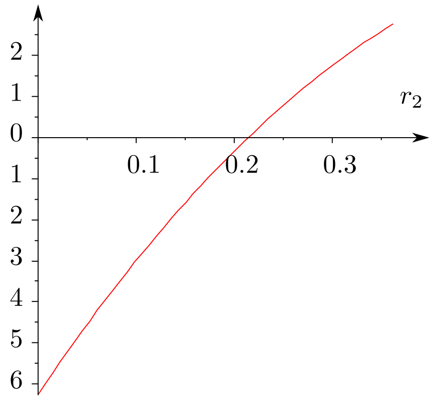


Figure C.3. $\triangle_P(r_2)$ with $\triangle_P(h_1h_5) > 0$.

We now present the loops corresponding to FIGs. C.1 and C.2. In the first case (FIG. C.1), we have $\triangle_P(r_2) = 0$ if $r_2 = r_{2,-} = 0.0807\ldots$ and $r_2 = r_{2,+} = 0.111\ldots$, and in the second case, we have $\triangle_P(r_2) = 0$ if $r_2 = r_{2,-} = 0.0821\ldots$ and $r_2 = r_{2,+} = 0.25$. Then, we allow r_2 to vary in $[r_{2,-}, r_{2,+}]$. The solutions of the polynomial $P(z)$ in (C16) make it possible to draw (see FIG. C.4) the triangle-shaped loop Γ in the space of roots (α, β, γ) corresponding to FIG. C.1.

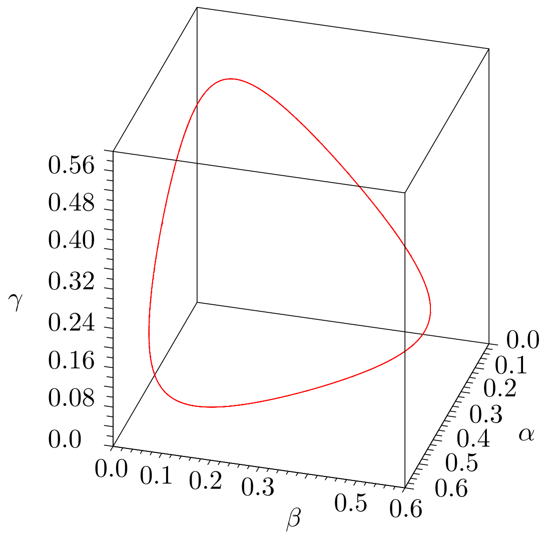


Figure C.4. The triangle-shaped loop Γ

Only one loop L in $\mathbb{R}^4$ corresponds to this loop Γ deduced from the relations (C13) which defines an embedding from $\mathbb{R}^3$ to $\mathbb{R}^4$. This loop L represents the nonunivocity mentioned in the introduction. However, contrary to appearances, the loop L (and Γ as a result) comprises six connected segments each obtained when varying r_2 in $[r_{2,-}, r_{2,+}]$ and connected at their two ends corresponding to the values $r_{2,\pm}$. In other words, we do not obtain a loop parameterized continuously by r_2 because we must count each of the six segments to fit them together into a loop. Indeed, from (C14) with the parameters r_i fixed, we obtain the following system of equations:

$$\alpha + \beta + \gamma = r_1 \,, \tag{C35a}$$

$$\alpha^2 + \beta^2 + \gamma^2 = r_1^2 - 2r_2 \, (> 0) \,, \tag{C35b}$$

$$\alpha\beta\gamma = r_3 \,. \tag{C35c}$$

However, (C35c) is the equation of a Tzitzeica surface,[48] which has the symmetry of the tetrahedron with the tetrahedral symmetry group $T_g \simeq S_4$ of order 24. It has four strata (or 'leaves') (see Fig. C.5). One stratum only, denoted by T_+, corresponds to the positive values for all coordinates α, β and γ. This stratum has a symmetry of rotation of order 3 along the axis generated by the vector $(1,1,1)$ and the plane $\alpha = \beta$ (or $\beta = \gamma$ or $\alpha = \gamma$) is a plane of reflection symmetry. Thus, for each leaf, the symmetry group is the dihedral group D_3 of order 6 of the equilateral triangle. Then, because T_g is a subgroup of the group of rotations

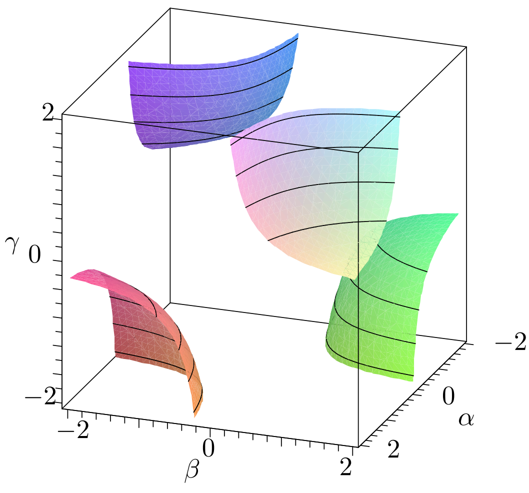


Figure C.5. The Tzitzeica surface

[48] A particular tetrahedral Goursat surface of degree 3 for which the general equation is: $x\,y\,z + a(x^2+y^2+z^2) = b$, with a, $b \in \mathbb{R}$.

of the sphere S^2, to each solution of the system of equations (C35b) and (C35c) corresponds, actually, a set of 23 other solution points obtained by application of the maps in T_g, each point on one of the four strata. Adding the equation (C35a) of the plane to this system restricts the set of solutions to T_+. Moreover, because the number of unknowns α, β and γ is equal to the number of equations, then, we must consider only one solution point as solution of the system of equations (C35b) and (C35c). Hence, by symmetry, we have only at most 6 solution points of the full system (C35) on the leaf T_+. This set remains invariant under the action of the normal sub-group $D_3 \subset T_g$ which also exchanges the strata of the Tzitzeica surface, in contrast to T_+. Obviously, D_3 corresponds isomorphically to the group of permutations $S_3 \simeq D_3$ of the three coordinates α, β and γ. Hence, because the sphere and the plane defined respectively by (C35a) and (C35b) are also invariant with respect to D_3, the number of solutions of (C35) is exactly 6, all on T_+. Then, varying r_2, the unique loops Γ or L comprise six "tracking" segments connected at their boundaries and contained in the planes of the three reflection symmetries of D_3. There are theorems [BCR98, § 13] on such solution loops (or spheres more generally) only for finite set of homogeneous polynomials (*i.e.*, *forms*) of same degrees (*e.g.*, Hopf fibrations, or Cayley–Dickson fibrations). Thus, this example, as a new result, suggests that a generalization to finite sets of homogeneous polynomials of different degrees is possible. Actually, on the basis of the discussion above, we have proved that

Theorem C.1. *Let S be the system of algebraic homogeneous equations*

$$\sum_{i=1}^{4} x_i = h_1 > 0\,, \qquad \sum_{i<j=1}^{4} x_i x_j = h_2 > 0\,, \qquad \prod_{i=1}^{4} x_i = h_4 > 0\,,$$

and h_5 and h_6 such that

$$h_5 \equiv \frac{1}{h_1}\left(\frac{1}{16}\left(h_1^2 - 4h_2\right)^2 - 4h_4\right) > 0\,, \qquad h_6 \equiv \frac{3}{4}h_1^2 - 2h_2 > 0\,.$$

If the polynomial $\triangle_P(r_2)$ has only 2 simple roots in the interval $\mathcal{D} \equiv \Big[0,\, \min\left(h_1 h_5,\, h_6^2/3\right)\Big]$, then the space of solutions of S is homeomorphic to S^1.

We conjecture that we can not have four positive roots in $\mathcal{D}$ and therefore that S can not be homeomorphic to $(S^1)^2$.

If we now trace the curve corresponding to case 3 (FIG. C.3), we no longer obtain a triangle-shaped loop but instead we obtain three small disjointed segments of curves that appear similar to kind of "cheeks" (FIG. C.6) placed at the "vertices" (corners) of a hypothetical triangle-shaped loop (FIG. C.7).

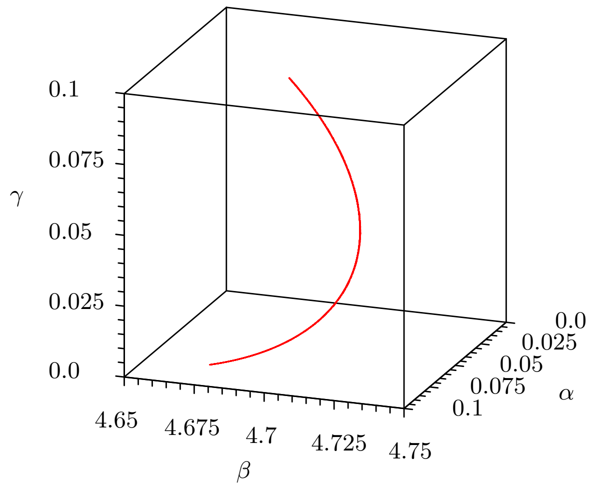


Figure C.6. A "cheek" at one of the three vertices

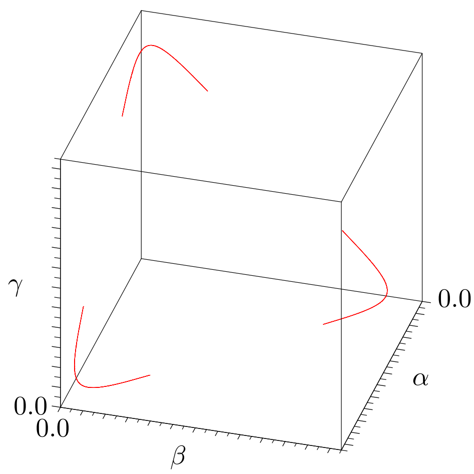


Figure C.7. The three "cheeks" at the vertices of a "hypothetic" triangle-shaped loop

Each cheek is a little curve made of two segments that are again connected in the planes of the reflection symmetries.

6 Conclusion

We have shown that the system of three polynomial equations (C8b), (C8c) and (C8d) in the variables ν_i^2 ($i = 1, \ldots, 4$) has a solution loop L. With respect to the variables β_k ($k = 1, \ldots, 4$), we also obtain a solution loop, but the variables β_k must also be solutions to the supplementary linear equation (C8a) which reduces the solution set to a finite set of isolated solution points. Thus, assuming that there exists only one traceless metric field $G_\perp$ in a given $\{ssst\}$-frame, $G_\perp$ corresponds to an infinite set, a loop, of metric fields G_{-1} in a given $\{\ell\ell\ell\ell\}$-frame. Obviously, the converse is not true, which means that we can argue based on mathematical arguments only that the $\{\ell\ell\ell\ell\}$-frames have to be physically prior to the $\{ssst\}$-frames. Hence, the latter can be deduced from the former and built up from physical light-like processes only, and we can then clarify the meaning of the expression "*make sense*" in Proposition 1 about space and time coordinates.

Appendix D: The Sturm sequence of $\triangle_P(r_2)$

We recall that Sturm's theorem allows computing the number of distinct roots of a univariate polynomial contained in a given interval, and then we can deduce, possibly, the signs of the polynomial in this interval [Coh93, Cos00]. For we need to compute the so-called Sturm sequence of, in particular, the polynomial $\triangle_P(r_2)$ in the variable r_2 in the interval $[0, h_1\, h_5]$. In this appendix, we show that the Sturm sequence we obtain gives tremendous polynomials in h_1 or h_5 from which we cannot deduce algebraically the number of roots. And thus, Sturm's method is not suitable in the present situation, justifying the other approach developed in the last appendix.

The Sturm sequence of a polynomial $P(z)$ is a sequence of polynomials $P_i(z)$ ($i = 0, \ldots, \deg P$) such that $P_0 \equiv P$, $P_1 \equiv P'$ (the first derivative of P) and $P_{i-2} \equiv P_{i-1}\, Q_{i-1} - P_i$ for $i = 2, \ldots, \deg P$, *i.e.*, the P_is and the Q_is are respectively the successive remainders and quotients from applying Euclid's algorithm to the division of polynomials. Then, to know the number of roots of $P(z)$ in the given interval $[a,\, b]$ where a and b are not roots of P, we have first to know the number $n(a)$ of sign changes in the sequence $\{P_0(a), \ldots,\, P_4(a)\}$, and the number $n(b)$ of sign changes in the sequence $\{P_0(b), \ldots,\, P_4(b)\}$. Then, the number of roots of P in the interval $[a,\, b]$ is $n(a) - n(b)$. Actually, we will show that we cannot compute the numbers $n(a)$ or $n(b)$ in the present situation.

Indeed, we start with the polynomial

$$P_0(z) \equiv -\frac{{h_1}^4}{27}\, \triangle_P(z) = -\frac{1}{27} \sum_{i=0}^{4} a_i\, z^i\,, \tag{D1}$$

where the a_is are given by the relations (C22). Then, denoting by b_i the coefficients such that $b_i \equiv -a_i/27$, we find the following polynomials for the Sturm sequence of P_0:

$$P_0(z) = z^4 + b_3\, z^3 + b_2\, z^2 + b_1\, z + b_0\,, \tag{D2a}$$

$$P_1(z) = 4\, z^3 + 3\, b_3\, z^2 + 2\, b_2\, z + b_1\,, \tag{D2b}$$

$$P_2(z) = \frac{1}{2}\left(\frac{3}{8}\, {b_3}^2 - b_2\right) z^2 + \frac{1}{4}\left(\frac{1}{2}\, b_2\, b_3 - 3\, b_1\right) z + \left(\frac{1}{16}\, b_1\, b_3 - b_0\right) \tag{D2c}$$

$$\equiv c_2\, z^2 + c_1\, z + c_0\,, \tag{D2d}$$

$$P_3(z) = -\frac{1}{{c_2}^2}\left(\left(4\, {c_1}^2 + 2\, b_2\, {c_2}^2 - 3\, b_3\, c_1\, c_2 - 4\, c_0\, c_2\right) z + \left(4\, c_0\, c_1 + b_1\, {c_2}^2 - 3\, b_3\, c_0\, c_2\right)\right), \tag{D2e}$$

$$\equiv d_1\, z + d_0\,, \tag{D2f}$$

$$P_4(z) = \frac{1}{{d_1}^2}\left(c_1\, d_0\, d_1 - c_2\, {d_0}^2 - c_0\, {d_1}^2\right). \tag{D2g}$$

We have to know, in particular, the number $n(0)$, and thus, we must compute the coefficients $P_i(0)$, *i.e.*, b_0, b_1, c_0, d_0 and $P_4(0)$ in terms of h_1 and h_5. We find the following expressions (not given in complete detail):

$$P_0(0) = b_0 = \frac{1}{27}\,{h_1}^4\,{h_5}^2\,(4\,{h_6}^3 + 27\,{h_5}^2) \geqslant 0\,, \tag{D3a}$$

$$P_1(0) = b_1 = -\frac{2}{27}\,{h_1}^3\,h_5\,(54\,{h_5}^2 + 9\,h_1\,h_5\,h_6 + 4\,{h_6}^3) \leqslant 0\,, \tag{D3b}$$

$$\begin{aligned} P_2(0) = c_0 = -\frac{1}{2916}\,{h_1}^4\,h_5\,(108\,{h_1}^3\,{h_5}^2 + 18\,{h_1}^4\,h_5\,h_6 + 8\,{h_1}^3\,{h_6}^3 \\ - 972\,h_6\,h_1\,{h_5}^2 - 81\,{h_6}^2\,{h_1}^2\,h_5 - 36\,{h_6}^4\,h_1 \\ + 216\,h_5\,{h_6}^3)\,, \end{aligned} \tag{D3c}$$

$$P_3(0) = d_0 = -A^2\,{h_1}^3\,h_5\,\Big({h_1}^9\,h_5\,(63\,{h_6}^3 + 486\,{h_5}^2) + \ldots\Big)\,, \tag{D3d}$$

$$P_4(0) = \ldots \tag{D3e}$$

where A is a fraction depending on h_1, h_5 and h_6, and where $P_4(0)$ is a polynomial expression of degree 24 in h_1. Clearly, we cannot compute algebraically in full generality the signs of these expressions and, furthermore, the number of roots in $[0, h_1\,h_5]$.

Appendix E: The univariate polynomial equations of degree 3 or 4 and their positive roots

1 – The discriminant and the roots of polynomial equations of degree 3

For polynomials $P(x) \in \mathbb{R}[x]$ of degree 3 such as $P(x) \equiv x^3 - \mu_1\,x^2 + \mu_2\,x - \mu_3$, with the discriminant

$$\triangle_3 \equiv \prod_{i<j=1}^{3} (x_i - x_j)^2 \equiv 18\,\mu_1\mu_2\mu_3 + {\mu_1}^2{\mu_2}^2 - 4\,{\mu_1}^3\mu_3 - 4\,{\mu_2}^3 - 27\,{\mu_3}^2 \tag{E1}$$

and the roots x_k $(k = 1, 2, 3)$, we have the following theorem:

Theorem E.1. – *Let $\triangle_3$ be the discriminant of the univariate polynomial $P(x) \in \mathbb{R}[x]$ of degree 3; then we have:*

1. *if $\triangle_3 < 0$, there are 1 real root and 2 complex conjugate roots,*
2. *if $\triangle_3 = 0$, there are 1 simple real root, 1 double real root or 1 triple real root, and*
3. *if $\triangle_3 > 0$, there are 3 simple real roots.*

Moreover, we recall that the μ_ks are the elementary symmetric polynomials of degree k in the roots x_h of $P(x)$ (*e.g.*, $\mu_2 \equiv \sum_{i<j=1}^{3} x_i\, x_j$).

2 – The discriminant and the roots of polynomial equations of degree 4

Let $P(z) \in \mathbb{R}[z]$ a polynomial of degree 4 such that

$$P(z) = z^4 - \sigma_1\, z^3 + \sigma_2\, z^2 - \sigma_3\, z + \sigma_4\,. \tag{E2}$$

We denote by z_i its real or complex roots ($i = 1, \ldots, 4$), and again, the coefficients σ_h are the elementary symmetric polynomials of degree h in the roots z_j. The 'discriminant' $\triangle_4$ of $P(z)$ is defined up to a positive constant as the product of the squares of the six differences between the roots z_i [GKZ94, §III. 12. pp. 397–425][Cre99, §4.1.1 p. 76]:

$$\triangle_4 \equiv \prod_{i<j=1}^{4} (z_i - z_j)^2\,. \tag{E3}$$

It can be expressed as a multivariate polynomial in the coefficients σ_k via the relation

$$\begin{aligned}\triangle_4 \equiv{}& 256\,{\sigma_4}^3 + \Big(144\,\sigma_2{\sigma_1}^2 - 27\,{\sigma_1}^4 - 128\,{\sigma_2}^2 - 192\,\sigma_1\sigma_3\Big)\,{\sigma_4}^2 \\ &+ 2\,\Big(72\,{\sigma_3}^2\sigma_2 - 3\,{\sigma_1}^2{\sigma_3}^2 - 2\,{\sigma_2}^3{\sigma_1}^2 - 40\,\sigma_1\sigma_3{\sigma_2}^2 + 9\,{\sigma_1}^3\sigma_3\sigma_2 \\ &\qquad + 8\,{\sigma_2}^4\Big)\,\sigma_4 + \Big(18\,\sigma_1\sigma_2\sigma_3 + {\sigma_2}^2{\sigma_1}^2 - 4\,{\sigma_1}^3\sigma_3 - 4\,{\sigma_2}^3 - 27\,{\sigma_3}^2\Big)\,{\sigma_3}^2\,.\end{aligned} \tag{E4}$$

Then, the well-known properties that $\triangle_4$ yields are the following:

Lemma E.1. – *Let $\triangle_4$ be the discriminant of the univariate polynomial $P(z) \in \mathbb{R}[z]$ of degree 4; then we have:*

1. *if $\triangle_4 < 0$, there are 2 real distinct roots and 2 complex conjugate roots,*
2. *if $\triangle_4 = 0$, there is a multiple root, and*
3. *if $\triangle_4 > 0$, there are 4 real distinct roots or 2 complex distinct roots with complex conjugates.*

Nevertheless, we must have other "discriminants/seminvariants" for quartic equations. To obtain these discriminants, we notice that what distinguishes the real numbers from the complex numbers is that the latter may have negative squares. Thus, linearly combining the roots z_i to exhibit possible imaginary parts, we can consider the following three variables:

$$\begin{aligned}
&k_1 = (z_1 + z_2 - z_3 - z_4)^2\,,\\
&k_2 = (z_1 - z_2 + z_3 - z_4)^2\,,\\
&k_3 = (z_1 - z_2 - z_3 + z_4)^2\,.
\end{aligned} \tag{E5}$$

These variables are permuted nontrivially under permutations in $S_4/D_2 \simeq S_3 \subset S_4$ where S_4 is the Galois group of permutations of the four roots of $P(z)$, and D_2 is the dihedral group of order (cardinality) 4, isomorphic to the abelian Klein four-group $\mathbb{Z}_2{\times}\mathbb{Z}_2$. Therefore, the k_js can be expressed as $\mathbb{C}$-algebraic expressions depending only on the coefficients of $P(z)$. Moreover, we see that the three variables Z_i where

$$\begin{aligned}
&Z_1 \equiv z_1\,z_2 + z_3\,z_4 = \frac{1}{4}(k_1 - \sigma_1^2 + 4\sigma_2)\,,\\
&Z_2 \equiv z_1\,z_3 + z_2\,z_4 = \frac{1}{4}(k_2 - \sigma_1^2 + 4\sigma_2)\,,\\
&Z_3 \equiv z_1\,z_4 + z_2\,z_3 = \frac{1}{4}(k_3 - \sigma_1^2 + 4\sigma_2)\,,
\end{aligned} \tag{E6}$$

are the roots of the so-called *cubic resolvent* of $P(z)$ — a polynomial of degree 3 associated with $P(z)$ — with $\triangle_4$ also as discriminant. These variables verify the well-known relations

$$\begin{aligned}
Z_1 + Z_2 + Z_3 &= \sigma_2\,,\\
Z_1Z_2 + Z_2Z_3 + Z_1Z_3 &= \sigma_1\sigma_3 - 4\sigma_4\,,\\
Z_1Z_2Z_3 &= \sigma_4\sigma_1^2 + \sigma_3^2 - 4\sigma_2\sigma_4\,.
\end{aligned} \tag{E7}$$

Because the k_is are deduced from the Z_is by the same translation and dilatation, they are roots of certain equations which again have the discriminant $\triangle_4$ (defined up to a positive factor 4^6). However, the difference occurs when computing the elementary symmetric polynomials K_h in the variables k_i. Indeed, we obtain:

$$
\begin{aligned}
K_1 &\equiv \sum_{i=1}^{3} k_i = 3\sigma_1^2 - 8\sigma_2\,, \\
K_2 &\equiv \sum_{i<j=1}^{3} k_i\, k_j = 3\sigma_1^4 + 16\left(\sigma_2^2 + \sigma_1\sigma_3 - \sigma_2\sigma_1^2 - 4\,\sigma_4\right), \\
K_3 &\equiv \prod_{i=1}^{3} k_i = \left(8\,\sigma_3 + \sigma_1(\sigma_1^2 - 4\,\sigma_2)\right)^2 \geqslant 0\,.
\end{aligned}
\tag{E8}
$$

In addition, in a framework illustrating the Tarski-Seidenberg theorem [BCR98, §1.4 pp. 17–22][Cos00, §1.3 pp. 10–17] , we define the '*Boolean sequence*' $BS_3(K)$ as follows:

Definition E.1. – *We call* 'Boolean sequence' $BS_3(K)$ *a particular Boolean combination generated from a recursive finite disjunction of systems of equations and inequalities with the* 3 *variables* K_ℓ $(\ell = 1, 2, 3)$ *such that*

$$
\begin{aligned}
&BS_3(K) = \Big\{K_3 > 0\,,\ \textit{and}\ K_2 > 0\,,\ \textit{and}\ K_1 > 0\Big\} \\
&\textit{or}\ \Big\{K_3 = 0\,,\ \textit{and}\ K_2 > 0\,,\ \textit{and}\ K_1 > 0\Big\} \\
&\textit{or}\ \Big\{K_3 = 0\,,\ \textit{and}\ K_2 = 0\,,\ \textit{and}\ K_1 > 0\Big\} \\
&\qquad\qquad \textit{or}\ \Big\{K_3 = 0\,,\ \textit{and}\ K_2 = 0\,,\ \textit{and}\ K_1 = 0\Big\}.
\end{aligned}
\tag{E9}
$$

Then, we have the following property:

Lemma E.2. – *If* $\triangle_4 \geqslant 0$*, then the parameters* K_i $(i = 1, 2, 3)$ *satisfy* $BS_3(K)$ *if and only if all of the roots* z_j *of the polynomial* $P(z) \in \mathbb{R}[z]$ *of degree 4 are real.*

Proof – Indeed, let $\varphi : Z \equiv (z_1, z_2, z_3, z_4) \longrightarrow U \equiv (u_1, u_2, u_3, u_4)$ be the map defined by the four linear relations

$$
\begin{aligned}
u_1 &\equiv z_1 + z_2 - z_3 - z_4\,, \\
u_2 &\equiv z_1 - z_2 + z_3 - z_4\,, \\
u_3 &\equiv z_1 - z_2 - z_3 + z_4\,, \\
u_4 &\equiv \sigma_1 = z_1 + z_2 + z_3 + z_4 \in \mathbb{R}\,,
\end{aligned}
\tag{E10}
$$

then, because φ is bijective ($\det\varphi = 16$), we obtain that $U \in \mathbb{R}^4 \iff Z \in \mathbb{R}^4$ or, equivalently, $k_j \geqslant 0\,(j = 1, 2, 3) \iff Z \in \mathbb{R}^4$. In other words, we find that $BS_3(k) \iff Z \in \mathbb{R}^4$. Besides, we obviously have that $BS_3(k) \implies BS_3(K)$, but also that $BS_3(K) \implies BS_3(k)$. Indeed, firstly, because $\triangle_4 \equiv \triangle_3 \geqslant 0$, then all of the roots k_i of a cubic equation are real, and then, applying Descartes's rule of signs to the polynomial

$$S(z) = z^3 - K_1\,z^2 + K_2\,z - K_3 \tag{E11}$$

of degree 3 with roots k_i, we deduce that all of the $K_i > 0$ if and only if all of the $k_j > 0$. And, secondly, if $K_3 = 0$, $K_1 > 0$ and $K_2 > 0$, then, for instance, $k_3 = 0$ and $K_2 = k_1 k_2 > 0$. And the two remaining cases $K_3 = 0$, $K_1 = 0$ and $K_2 > 0$, and $K_3 = 0$, $K_1 = 0$ and $K_2 = 0$ involve trivially similar relationships for the k_js. Therefore, we deduce that $BS_3(k) \iff BS_3(K)$; hence the result. □

As a result, we obtain the following obvious corollary:

Corollary E.1. – *If $\triangle_4 > 0$ and the parameters K_i $(i = 1, 2, 3)$ do not satisfy $BS_3(K)$, then all of the roots z_j of the polynomial $P(z) \in \mathbb{R}[z]$ of degree 4 are complex.*

Now, we must investigate the case in which $\triangle_4 = 0$ with at least one $K_j \leqslant 0$ and $BS_3(K)$ not satisfied. It is important to note that 1) $\triangle_S \equiv \triangle_4$, where $\triangle_S$ is the discriminant of the polynomial $S(z)$ defined by the relation (E11), 2) the variables K_i are all real, and 3) one of the k_j is real ($S(z)$ is of degree 3).

Moreover, if one of the K_j is nonpositive and a double root z_k exists, *i.e.*, $\triangle_4 = 0$, then first, a double root k_j exists ($\triangle_S = 0$), and second, a negative root k_i exists; otherwise, all the K_j would be positive. For example, we can take $k_2 = k_3$ and then $K_3 = k_1 k_2^2$. Hence, k_1 and k_2 are real because $K_3 \in \mathbb{R}$, and if $K_3 \neq 0$, then, the double root k_2 is negative and, moreover, the simple root k_1 is positive because we have always $K_3 \geqslant 0$. Thus, we obtain the following:

Theorem E.2. – *If $\triangle_4 = 0$ and $K_3 > 0$, and if $K_1 \leqslant 0$ or $K_2 \leqslant 0$, then the polynomial $P(z)$ of degree 4 has a double real root and a complex root with its complex conjugate.*

Proof – The assumptions regarding the K_is involve $k_1 > 0$ and $k_2 = k_3 < 0$ to cancel out a K_i, or makes it negative. From $k_2 = k_3$, we obtain 1) $z_3 = z_4$, or 2) $z_1 = z_2$. And then, because $k_2 < 0$, we have 1) $z_3 = z_4$ and $z_2 = \bar{z}_1$, or 2) $z_1 = z_2$ and $z_4 = \bar{z}_3$. □

In contrast, if $K_3 = 0$, and defining the three variables H_i $(i = 1,\, 2,\, 3)$ such that

$$\begin{aligned} H_1 &\equiv 4K_2 - K_1^2\,, \\ H_2 &\equiv \sigma_1^2(\sigma_1^4 - 256\,\sigma_4) = \sigma_1^2\left(2K_1\sigma_1^2 + H_1 - 8\,\sigma_1\varepsilon\sqrt{K_3}\right)\,, \\ H_3 &\equiv \sigma_1(\sigma_1^3 - 16\,\sigma_3) = \sigma_1\left(\sigma_1 K_1 - 2\,\varepsilon\sqrt{K_3}\right)\,, \end{aligned} \tag{E12}$$

where the relationships between the H_ks and the K_is are obtained both for one particular $\varepsilon = \pm 1$ only, then we get another theorem whenever $K_3 = 0$:

Theorem E.3. – *If $\triangle_4 = 0$, $K_3 = 0$ and $\sigma_4 \neq 0$, then*

1. *if $H_1 = 0$ (and then $K_2 \geqslant 0$), we have a double nonvanishing real root $z_3 = \sigma_1/4$ and two other nonvanishing roots z_1 and z_2 such that*
   (a) *if $H_2 > 0$, z_1 and z_2 are two distinct real simple roots,*
   (b) *if $H_2 = 0$, $z_1 = z_2 = z_3 \in \mathbb{R}^*$,*
   (c) *if $H_2 < 0$, $z_2 = \bar{z}_1 \in \mathbb{C} - \mathbb{R}$ and $z_3 = \Re e(z_1)$, and*
2. *if $H_1 \neq 0$ (and then $K_2 = 0$), we have two double nonvanishing roots z_1 and z_2 such that*
   (a) *if $\sigma_1 \neq 0$, then*
      i. *if $H_3 > 0$, z_1 and z_2 are two distinct real roots,*
      ii. *if $H_3 = 0$, $z_1 = z_2 \in \mathbb{R}^*$,*
      iii. *if $H_3 < 0$, $z_2 = \bar{z}_1 \in \mathbb{C} - \mathbb{R}$, and*
   (b) *if $\sigma_1 = 0$, then*
      i. *if $\sigma_2 > 0$, $z_2 = \bar{z}_1 = -z_1 = -i\sqrt{\sigma_2/2} \in i\,\mathbb{R}^*$, and*
      ii. *if $\sigma_2 < 0$, $z_1 = -z_2 = \sqrt{\sigma_2/2} \in \mathbb{R}^*$.*

Proof – The relation $\triangle_4 \equiv \triangle_S = 0$ involves that we have a double root k_i; for instance, we take $k_2 = k_3$, and therefore, $z_1 = z_2$, or, symmetrically, $z_3 = z_4$. For the sake of simplicity and convenience, we can take $z_3 = z_4$. If $K_3 = k_1 k_2^2 = 0$, then 1) $k_1 = 0$, or 2) $k_1 \neq 0$ and $k_2 = k_3 = 0$. Thus, we have the following:

- If $k_1 = 0$ and $z_3 = z_4$, then $z_3 = z_4 = (z_1 + z_2)/2$. Thus, this case appears if and only if $H_1 \equiv 4K_2 - K_1^2 = 0$ and $K_3 = 0$. It follows that $\sigma_1 = 2(z_1 + z_2)$ and $\sigma_4 = z_1 z_2 (z_1 + z_2)^2/4$. Then, because $\sigma_4 \neq 0$, we obtain $\sigma_1 \neq 0$. Therefore, we have $z_1 + z_2 = \sigma_1/2$, $z_1 z_2 = 16\,\sigma_4/\sigma_1^2$ and $z_3 = \sigma_1/4 \in \mathbb{R}^*$. Hence, z_1 and z_2 are the roots of the polynomial $2\sigma_1^2\, x^2 - \sigma_1^3\, x + 32\sigma_4$ with the discriminant $H_2 \equiv \sigma_1^2(\sigma_1^4 - 256\,\sigma_4)$.

- If $k_1 \neq 0$, $k_2 = k_3 = 0$ and $z_3 = z_4$, then $z_1 = z_2$ as well. This case appears if and only if $K_2 = 0$ and $K_3 = 0$. Then, we have $H_1 = -k_1^2 \neq 0$. In particular, if $\sigma_1 \neq 0$, we obtain

$$z_1 + z_3 = \frac{1}{2}\,\sigma_1\,, \qquad\qquad z_1 z_3 = \frac{\sigma_3}{\sigma_1}\,. \tag{E13}$$

  Hence, z_1 and z_3 are the roots of the polynomial $2\sigma_1\, x^2 - \sigma_1^2\, x + 2\sigma_3$ with the discriminant $H_3 \equiv \sigma_1(\sigma_1^3 - 16\,\sigma_3)$, which determines if the roots are real or complex conjugates. If $\sigma_1 = 0$, then $z_1 = -z_3$, $\sigma_3 = 0$ and $\sigma_2 = -2z_1^2 \in \mathbb{R}^*$. Therefore, $z_1 \in \mathbb{R}^*$ or $i\,\mathbb{R}^*$, depending on the sign of σ_2.

□

Additionally, we would like to know the signs of the real roots of $P(z)$.

3 – The signs of the real roots

For polynomial equations of degree 3, we have the following:

Theorem E.4. – *Let $P(x)$ be the polynomial of degree 3 such that $P(x) \equiv x^3 - \mu_1\, x^2 + \mu_2\, x - \mu_3 \in \mathbb{R}[x]$, assuming $\mu_3 \neq 0$. Let L be such that $L \equiv 27\mu_3 - \mu_1^3$; then, the signs of the roots of the polynomial $P(x)$ of degree 3 defined above are the following: If $\triangle_3$ is the discriminant of $P(x)$, then*

1. *if $\triangle_3 < 0$, the unique simple real root has the sign of μ_3,*

2. *if $\triangle_3 > 0$, there are three distinct simple real roots, and if $\mu_2 > 0$ and $\mu_1\mu_3 > 0$, then the roots must all have the same sign equal to $\operatorname{sgn}(\mu_3)$; otherwise, if $\mu_2 \leqslant 0$ or $\mu_1\mu_3 \leqslant 0$, only one root has the sign of μ_3, and*

3. *if $\triangle_3 = 0$, then*

   (a) *if $L = 0$, the sign of the unique triple real root is the sign of μ_3, and*

   (b) *if $L \neq 0$, there are two distinct real roots. The simple root has the sign of μ_3. The double root has the same sign as the simple root if and only if $\mu_2 > 0$ and $\mu_1\mu_3 > 0$.*

Proof – If $\triangle_3 < 0$, then it is well-known that there is only one real root x_1 and that there are two complex conjugate roots x_2 and $\bar{x}_2$. Thus, $\mu_3 \equiv x_1|x_2|^2$ $(\neq 0)$, and $\mathrm{sgn}(x_1) = \mathrm{sgn}(\mu_3)$.

If $\triangle_3 > 0$, then there are three nonvanishing real distinct roots x_k $(k = 1, 2, 3)$. Then, applying Descartes's rule of signs, we see that $\mu_2 > 0$ and $\mu_1\mu_3 > 0$ if and only if all of the nonvanishing real roots have the same sign and the sign of μ_3.

Lastly, if $\triangle_3 = 0$, then all of the roots are real, and there are two exclusive alternatives: one triple root x_3 or one simple root x_1 and a double root x_2. In the former case, we have $\mu_1 = 3x_3$ and $\mu_3 = x_3^3$, *i.e.*, $L = 0$, and then, $\mathrm{sgn}(x_3) = \mathrm{sgn}(\mu_1) = \mathrm{sgn}(\mu_3)$. In the last case, because $\mu_3 = x_1x_2^2$, then $\mathrm{sgn}(x_1) = \mathrm{sgn}(\mu_3)$ and $L \neq 0$. However, applying Descartes's rule of signs again, we know that the roots have the same signs if and only if for $\varepsilon = +1$ or $\varepsilon = -1$, all of the $\varepsilon^i\mu_i \geqslant 0$ $(i = 1, 2, 3)$. Thus, we have $\mu_1\mu_3 > 0$ and $\mu_2 > 0$. $\square$

For polynomial equations of degree 4, we have the following last theorem:

Theorem E.5. – *Let $P(z)$ be the polynomial of degree 4 such that $P(z) = z^4 - \sigma_1 z^3 + \sigma_2 z^2 - \sigma_3 z + \sigma_4$, assuming $\sigma_4 \neq 0$. Then, using the notations for the theorem E.3, and given that $P(z)$ admits nonvanishing real roots, we have:*

1. *if $\triangle_4 < 0$, then, we have only two simple real roots z_1 and z_2 such that $\mathrm{sgn}(z_1) = \mathrm{sgn}(\sigma_4)\,\mathrm{sgn}(z_2)$, and if $\sigma_4 > 0$, then $\mathrm{sgn}(z_1) = \mathrm{sgn}(\sigma_1 + \sigma_3)$,*

2. *if $\triangle_4 > 0$, then, we have four simple real roots z_i $(i = 1, \ldots, 4)$ such that*

   (a) *if $\sigma_4 > 0$, then $\mathrm{sgn}(z_i) = \mathrm{sgn}(\sigma_1)$ if $\sigma_1\,\sigma_2\,\sigma_3 > 0$, otherwise two roots are positive and the two others roots are negative,*

   (b) *if $\sigma_4 < 0$, then, $\mathrm{sgn}(z_1) = -\mathrm{sgn}(z_i) = -\varepsilon$ for all $i \neq 1$ where $\varepsilon = -\mathrm{sgn}(\sigma_3)$ if $\sigma_2 \leqslant 0$, and $\varepsilon = \mathrm{sgn}(\sigma_1)$ if $\sigma_2 > 0$,*

3. *if* $\triangle_4 = 0$ *and* $P(z)$ *admits real roots, then, we have*

   (a) *if* $K_3 = H_1 = 0$*, we have one double real root* z_3*, and two simple roots* z_1 *and* z_2 *such that*

      i. *if* $H_2 > 0$*, then,* $\operatorname{sgn}(z_1) = \operatorname{sgn}(\sigma_4)\operatorname{sgn}(z_2) = \operatorname{sgn}(z_3) = \operatorname{sgn}(\sigma_1)$*,*

      ii. *if* $H_2 = 0$*, then, the three roots are equal and* $\operatorname{sgn}(z_i) = \operatorname{sgn}(\sigma_1)$ $(i = 1, 2, 3)$*,*

      iii. *if* $H_2 < 0$*, then, only* z_3 *is real and* $\operatorname{sgn}(z_3) = \operatorname{sgn}(\sigma_1)$*,*

   (b) *if* $K_3 = 0$ *and* $H_1 \neq 0$*, we have two double real roots* z_1 *and* z_2 *such that*

      i. *if* $\sigma_1 \neq 0$*, then*

         A. *if* $H_3 > 0$*,* $z_1 \neq z_2$ *and* $\operatorname{sgn}(z_1) = \operatorname{sgn}(z_2) = \operatorname{sgn}(\sigma_1)$ *if* $\sigma_1\sigma_3 > 0$*, otherwise* $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_2)$*,*

         B. *if* $H_3 = 0$*,* $z_1 = z_2$ *and* $\operatorname{sgn}(z_1) = \operatorname{sgn}(\sigma_1)$*, and*

      ii. *if* $\sigma_1 = 0$ *and* $\sigma_2 < 0$*, then* $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_2)$*.*

Proof – We denote by c_+ (*resp.* c_-) the maximal number of positive (*resp.* negative) roots with their multiplicities taken into account. We investigate the following three cases ($\varepsilon = \pm 1$):

If $\triangle_4 < 0$, then the roots z_1 and z_2 are nonvanishing reals ($\sigma_4 \neq 0$), and $z_4 = \bar{z}_3 \in \mathbb{C} - \mathbb{R}$. Therefore, $\sigma_4 = z_1 z_2 |z_3|^2$, and thus, we deduce that $\operatorname{sgn}(z_1) = \operatorname{sgn}(\sigma_4)\operatorname{sgn}(z_2)$. Furthermore, if $\sigma_4 > 0$, the two real roots have the same sign. Therefore, necessarily, we have only $c_+ = 0$ or (exclusively) $c_- = 0$. Using Descartes's rule of signs, we have to check the number of sign changes, *i.e.*, c_ε, in the ordered sequence $S_\varepsilon^+ \equiv (+1, -\varepsilon\operatorname{sgn}(\sigma_1), \operatorname{sgn}(\sigma_2), -\varepsilon\operatorname{sgn}(\sigma_3), +1)$. Then, $c_\varepsilon = 0$ if and only if $(-\varepsilon)^i \sigma_i \geqslant 0$ for all $i \in \{1, 2, 3\}$, and $\operatorname{sgn}(z_k) = -\varepsilon$ for $k = 1, 2$. And then, we have always $\sigma_2 \geqslant 0$ and $\sigma_1\sigma_3 \geqslant 0$, and moreover, $\triangle_4 = 16\,\sigma_4(4\sigma_4 - \sigma_2^2)^2 \geqslant 0$ if $\sigma_1 = \sigma_3 = 0$. Hence, we deduce that $\operatorname{sgn}(z_k) = \operatorname{sgn}(\sigma_1 + \sigma_3) \neq 0$.

If $\triangle_4 > 0$, then we have four distinct nonvanishing real roots z_k if $\sigma_4 \neq 0$ and $K_i > 0$ for $i = 1, 2, 3$. It follows that we have only the three following alternatives, each exclusive: 1) $c_+ = 0$ or 3, 2) $c_- = 0$ or 3, and 3) $c_+ = c_- = 2$.

- If $c_\pm = 0$, *i.e.*, the four roots all have the same sign, then we have $\sigma_4 > 0$. However, again using Descartes's rule of signs, we have to consider again only the ordered sequences S_ε^+ for c_ε. Therefore, $c_\varepsilon = 0$ if and only if $(-\varepsilon)^i \sigma_i \geqslant 0$ for all $i = 1, 2, 3$, and $\operatorname{sgn}(z_k) = -\varepsilon$. And then, because the four nonvanishing roots all have the same sign, we have $\sigma_2 > 0$, $\sigma_1\,\sigma_3 > 0$ and then $\sigma_1\,\sigma_2\,\sigma_3 > 0$. Therefore, we have $\operatorname{sgn}(z_k) = \operatorname{sgn}(\sigma_1) \neq 0$.

- Second, if $c_\varepsilon = 3$, then $\sigma_4 < 0$ and we have the following two ordered sequences $S_\varepsilon^- \equiv (+1, -\varepsilon \operatorname{sgn}(\sigma_1), \operatorname{sgn}(\sigma_2), -\varepsilon \operatorname{sgn}(\sigma_3), -1)$. Then, evaluating at the 27 possible sequences S_ε^- when varying the $\operatorname{sgn}(\sigma_k)$ $(= \pm 1, 0\ ;\ k = 1, 2, 3)$, we deduce based on Descartes's rule of signs that $c_\varepsilon = 3$, *i.e.*, $\operatorname{sgn}(z_1) = -\operatorname{sgn}(z_j) = -\varepsilon$ $(j = 2, 3, 4)$ if and only if the ordered sequence $(-\varepsilon \operatorname{sgn}(\sigma_1),\ \operatorname{sgn}(\sigma_2),\ -\varepsilon \operatorname{sgn}(\sigma_3)) = (-, 0, +)$ or contains the ordered sub-sequence $(-, +)$. Then, if $\sigma_2 \leqslant 0$, we have the ordered sub-sequences $(-, 0, +)$ and $(\ldots, -, +)$ and then $-\varepsilon\sigma_3 > 0$, and if $\sigma_2 > 0$, then we have the ordered sub-sequences $(-, +, \ldots)$ and then $-\varepsilon\sigma_1 < 0$.

- Lastly, if $c_\pm = 2$, we again have $\sigma_4 > 0$, but we must also consider the sequence S_{+1}^+ only; hence, from the two precedent cases, we deduce that $\sigma_1\,\sigma_2\,\sigma_3 \leqslant 0$ whenever $c_\pm = 2$ and $\sigma_4 > 0$.

If $\triangle_4 = 0$, then, from the theorem E.3, we essentially have two cases to prove:

- $K_3 = H_1 = 0$.
  - If $H_2 > 0$, then, $\sigma_1 \neq 0$ and we have two simple roots z_1 and z_2, and one double root z_3. Thus, we obtain $\sigma_4 = z_1 z_2 z_3^2 \neq 0$. In addition, based on the proof of the theorem E.3, the relation $z_3 = \sigma_1/4$ holds. Therefore, $\operatorname{sgn}(z_3) = \operatorname{sgn}(\sigma_1)$ and $\operatorname{sgn}(z_1) = \operatorname{sgn}(\sigma_4)\operatorname{sgn}(z_2)$. In addition, because $z_1 + z_2 = \sigma_1/2$, then z_1 or z_2 must have the sign of σ_1.
  - If $H_2 = 0$, because we have only a quadruple real nonvanishing root $(\sigma_4 \neq 0)$, then $\sigma_1 = 4z_1 \neq 0$.
  - If $H_2 < 0$, then, $\sigma_1 \neq 0$ and $z_4 = z_3 = \Re e(z_1) = (z_1 + \bar{z}_1)/2 \in \mathbb{R}^*$, $z_2 = \bar{z}_1 \in \mathbb{C} - \mathbb{R}$ and $\sigma_4 = z_3^2 |z_1|^2 > 0$. Therefore, $\sigma_1 = 4z_3$ and then $\operatorname{sgn}(z_3) = \operatorname{sgn}(\sigma_1) \neq 0$.

- $K_3 = 0$, $H_1 \neq 0$ and $H_3 \geqslant 0$. Then, we have $\sigma_1 \neq 0$.

  On the basis of the proof of the theorem E.3, we have two double real roots $z_\pm$ such that

  $$z_\pm = \frac{1}{4\sigma_1}\left(\sigma_1^2 \pm \sqrt{H_3}\right) = \frac{\sigma_1}{4}\left(1 \pm \sqrt{1 - 16\,\frac{\sigma_3}{\sigma_1^3}}\right).$$

  Thus, depending on the sign of σ_3/σ_1^3 in the relation above, we deduce that $\operatorname{sgn}(z_\pm) = \operatorname{sgn}(\sigma_1)$ if $\sigma_1\sigma_3 > 0$ or $H_3 = 0$, otherwise $\operatorname{sgn}(z_+) = -\operatorname{sgn}(z_-) = \operatorname{sgn}(\sigma_1)$.

□

Appendix F: The space and time "projective" splitting

This appendix is in controversy... To be convinced of the projective structure in Einsteinian relativity, we begin within the context of special relativity and with the so-called '*time dilatation* phenomena.'

We know this phenomena is expressed by the following formula: $T \equiv \gamma\, T_0$, where T_0 is a time duration of a given phenomena in a rest frame and T the time duration of a same phenomena occurring in the moving frame, and observed in the rest frame. But, this formula is, somehow, in contradiction with the relativistic Doppler shift formula. Indeed, contrary to the relativistic Doppler shift formula, *i.e.*, $T = T_0\,\gamma\,(1+\beta\,\cos(\theta))$ (with $\beta = v/c$), the time dilatation formula does not depend on the direction of motion of the moving frame. Actually, we obtain the same formula only for the transversal relativistic Doppler shift, *i.e.*, when $\cos(\theta) = 0$. In particular, it is the time dilatation formula which is used in the famous twin paradox, and that the twins get closer or move away, the used formula is the same, *i.e.*, the time dilatation formula. Actually, we must use the relativistic Doppler shift formula which is sensitive to the direction of motion with $\theta = 0$ or π. And then, the result for a round trip is again given by $T = \frac{T_0}{2}\,\gamma\,(1+\beta) + \frac{T_0}{2}\,\gamma\,(1-\beta) \equiv T_0\,\gamma$, *i.e.*, the time dilatation formula for a round trip.

But then, what must be the physical interpretation of the time dilatation formula if this formula is insensitive to the direction of motion? That means it is about a phenomena insensitive to the direction of motion, and this cannot be ascribed to the spacetime 4-positions, say (x, y, z, t) in a Cartesian frame, of the events, and, in particular, to the set of 4-positions of a time-like worldline. In other words, the time dilatation formula applies only to vector lines of which the vector (x, y, z, t) is a generator, and not to affine point (x, y, z, t). Hence, this formula gives the change of "slope" of vector lines. Also, it means that special relativity deals only with (co-)vectors, (co-)tensors or vector lines, and not with affine geometrical objects such as a 4-position.

It is also very specific to the 4-velocities. The latter are defined from space-like 3-velocity vectors $\vec{v}$ and we consider that the corresponding 4-velocities are $(\gamma\vec{v}, \gamma c)$. But, it can be also the "4-velocity" $(-\gamma\vec{v}, -\gamma c)$ because the transformation law for the 3-velocities, associated with a Lorentz transformation applied to the corresponding 4-velocities, is a conformal transformation of the 3-velocity vectors $\vec{v}$; and this conformal transformation is the action of the projective group $PGL(4, \mathbb{R})$ insensitive to the signs of the directions.

Hence, special relativity is only about projective (co-)tensors, or "spacetime perspectives," and no affine objects such as 4-positions must intervene. These 4-positions can be taken in consideration, a priori, only within the framework of general relativity but with the condition to have a relativistic location system at disposal because general relativity appears to be only a generalization of the previously suggested, perspective viewpoint of the special relativity. In the lack of such system, we can only consider tensorial objects in the tangent Minkowski spacetimes and never events themselves. Also, it means that the physical interpretations are possible only after restoring the time and space splitting. It expresses the three-dimensional projective structure of the 3-velocity vectors which must be dealt with in the non-relativistic Galilean geometry associated with our observations.

Then in particular, the notion of proper time must be redefined within the context of the 4-dimensional conformal geometry (see [Rub10]).

Appendix G: Proof of the 1947 Ehresmann's Theorem

One of the most important Ehresmann's theorem utilizing the flow-box theorem in its proof (called "*Théorème de redressement de champs de vecteurs*" in french) is the following theorem stated for the first time in 1947 (Proposition 1, in: C. Ehresmann, «*Sur les espaces fibrés* *différentiables*», C.R. Acad. Sci. Paris, **224** (1947), pp. 1611–1612)[Ehr47a]:

> « Proposition 1. *Si E est compact, toute application différentiable p, en tout point de rang n, de E sur une variété B de dimension n détermine sur E une structure d'espace fibré différentiable.* »[49]

This statement differs from the one given three years later in a Bourbaki seminar in march 1950 and which was renewed in a colloquium on topology at Bruxelles: C. Ehresmann, "*Les connections infinitésimales dans un espace fibré différentiable*," Séminaire N. Bourbaki (March 1950), 1948–1951, exp. n° 24, pp. 153–168, p.154 [Ehr50]; Or also, C. Ehresmann, *Les connections infinitésimales dans un espace fibré différentiable*, Proceedings of the "Colloque de Topologie (espaces fibrés)," Centre Belge de Recherches Mathématiques (CBRM), 5–8 june 1950, Bruxelles, pp. 29–55, p.31 [Ehr51].

Outstandingly, in this 1950 version, Ehresmann refers to his article of 1947 published in the "Comptes Rendus de l'Académie des Sciences" (CRAS) while the statement is different. Indeed, in the first 1947 version the map p is only of class C^1 whereas it is of class C^2 in the 1950 version. In these first two versions the proof is never indicated. The statement given in 1950 is the following [Ehr50]:

> « Proposition. *Soient E et B deux variétés deux fois différentiables, B étant connexe, et soit p une application deux fois différentiable de E sur B, en tout point de rang égal à $\dim B$. Si E est compact ou bien si $p^{-1}(x)$ est compact connexe quel que soit $x \in B$, les ensembles $p^{-1}(x)$ sont les fibres d'une structure fibrée deux fois différentiable.* »[50]

Actually, in the 1950 version, Ehresmann needs the class C^2 because, in this particular paper on '*generalized spaces*,' he considered rings of exterior differential k-forms (necessarily defined on C^1-manifolds) and in particular integrable manifolds obtained from sets of completely integrable Pfaffian systems of 1-forms. And thus, the C^2 condition must be satisfied to verify the Frobenius's conditions (obtained from the differentiation of given differential 1-forms).

[49] "Proposition 1. — *If E is compact, any differentiable map p, everywhere of rank n, from E onto a manifold B of dimension n determines on E a differential fiber bundle structure.*"

[50] "Proposition – Let E and B be two twice differential manifolds, B connected, and let p be a twice differentiable map from E onto B, at every point of rank equal to $\dim B$. If E is compact or if $p^{-1}(x)$ is a compact connected set for all $x \in B$, then the sets $p^{-1}(x)$ are the fibers of a twice differential fibered structure."

To add to the problem, in the book: C. Godbillon, "*Feuilletages: études géométriques*," Progress in Mathematics, vol. **98**, Birkhäuser Verlag, Basel, 1991, C. Godbillon gives a modified version of the theorem [God91, p.16]:

> « **2.11 Théorème.** *Soient E et M deux variétés différentiables connexes sans bord. Une submersion $\pi : E \longrightarrow M$ propre et de classe C^r, $r \geqslant 2$, est une fibration localement trivialisable.* »[51]

C. Godbillon gives a proof which is, this time, unfortunately, partially incorrect although the general schema of the proof is right! Moreover, Godbillon adds, just after the statement of the theorem, that the map π is surjective, and consequently, that the statement must be modified substituting the word 'surmersion' for the word 'submersion.' Then, the expression "*fibration localement trivialisable*" (locally trivializable fibration) means exactly the same as "*fibré*" (fiber) without the adjective "*différentiable*" (differentiable).

Actually, a fibration is not always locally trivializable and, moreover, it is not always defined from a differentiable map. For instance, a fiber $E_m \subset E$ over a point $m \in M$ can intertwine infinitely around a point $e \in E$ in such a way that for all open U_e of e we obtain always $U_e \cap E_m \neq \varnothing$ and non-connected (as for instance the famous Hopf fibration non-locally trivializable). But, within the present context where π is differentiable, replacing "*fibration localement trivialisable*" by "*fibré différentiable*" would be better as indicated in the two Ehresmann's versions; all the more so as it involves homotopically equivalent fibers which is a situation not always satisfied for any fibration even, a priori, if the latter is locally trivializable.

In addition, the base space (B or M) is always connected, but Ehresmann did not assume that E is connected. Actually, in the Godbillon's proof, the need for E connected does not appear!

Lastly, in the statement of the theorem given by Godbillon, the assumption that π is proper is necessarily satisfied if E is compact as posed in the Ehresmann's statement. Indeed, a map is proper if the inverse image of a compact is a compact (to compare to the two assertions: 1) the image of a closed set by a closed map is a closed set, and 2) the inverse image of a closed

[51] **2.11 Theorem.** *Let E and M be two connected, differential manifolds without boundaries. A proper submersion $\pi : E \longrightarrow M$ of class C^r, $r \geqslant 2$, is a locally trivializable fibration.*

set by a continuous map is a closed set). On the other hand, the connexity of the fibers is not necessary as besides proven by Godbillon concerning this theorem. The fibers are de facto compact since they are the inverse images of always compact singletons by a proper map π.

Finally, the correct statement (we call "Ehresmann's theorem of completeness" for reasons given soon in the sequel) is the following:

> "**Ehresmann's theorem of completeness:** *Let E and M be two differential manifolds without boundaries with M connected. A proper surmersion $\pi : E \longrightarrow M$ of class C^r, $r \geqslant 1$, determines a differential fiber bundle structure on E.*"

Then, the proof is the following:

Proof.

$*$ Let $x_0 \in M$ and $F_{x_0} \equiv \pi^{-1}(x_0) \subset E$.

$*$ Let $C_{x_0} \equiv (V, (z_1, \ldots, z_m))$ where $V \subset M$ be a local chart on the C^r-manifold M such that $\dim M = m$ $(r \geqslant 1)$.

$*$ We assume that V is a neighborhood of x_0, *i.e.*, there exists an open $W \subset V$ such that $x_0 \in W$.

$*$ The map π is proper and thus F_{x_0} is compact.

$*$ M is locally compact (as for any topological manifold homeomorphic to $\mathbb{R}^m$, itself locally compact), and thus, we take V such that $\overline{V}$ is compact.

$*$ Let U be an open such that $U \subset \pi^{-1}(V)$. Then, U is relatively compact because 1) $\pi(U) \subset V$, and 2) π is continuous, and thus, we have $\pi(\overline{U}) \subset \overline{\pi(U)}$. Consequently, from 1) and 2) we deduce that $\pi(\overline{U}) \subset \overline{V}$ or, equivalently, that $\overline{U} \subset \pi^{-1}(\overline{V})$. Hence, since π is proper and $\overline{V}$ is compact it follows that $\overline{U}$ is compact, and thus, U is indeed relatively compact.

$*$ Moreover, we take U such that $W \subset \pi(U) \subset V$ and $F_{x_0} \subset U$.

$*$ Also, we take the chart C_{x_0} so that $\forall x \in V$ then $z_i(x) \neq 0$ for all of the indices $i = 1, \ldots, m$ (here is one of the most important differences with the proof of Godbillon in which the assumption is made, on the contrary, that to each point x_0 it corresponds values $z_i(x_0)$ all vanishing, and thus, to each x_0 it corresponds $0 \in \mathbb{R}^m$; and this raises a huge problem thereafter to apply the flow-box theorem). This choice of chart is available since V is relatively compact and because in the contrary case a translation would be carried out in $\mathbb{R}^m$ with, possibly, a dilatation to obtain a new open V containing no point in correspondence with the origin O of $\mathbb{R}^m$, and thus, equivalently, such that $z^{-1}(O) \notin V$.

$*$ Let $Z_i \in TE$ $(i = 1, \ldots, m)$ be m linearly independent vector fields with their supports in the open $U \subset E$ (relatively compact) such that on $U \cap \pi^{-1}(W)$ we have:

$$\pi_*(Z_i) \equiv \frac{\partial}{\partial z_i}$$

in W (relatively compact open).

$*$ Let $x \in W$, we denote by $\varphi(f, t)$ the flow of the vector field Z such that

$$Z \equiv \sum_{i=1}^{m} z_i \, Z_i$$

on U with the initial condition $f \in U \cap \pi^{-1}(W) \subset E$ at $t = 0$ and such that $\pi(f) = x$. Then, we obtain

$$\pi_*(Z) = \sum_{i=1}^{m} z_i \, \frac{\partial}{\partial z_i} \, ,$$

or, equivalently, the relation

$$\frac{dz_i}{dt} = z_i \, .$$

Thus, we find the solutions: $z_i(t) = z_i(0)\, e^t$ where $x \equiv (z_1(0), \ldots, z_m(0)) = \pi(f)$.

$*$ Hence, Z is non-singular, *i.e.*, regular on $U \cap \pi^{-1}(W) \subset E$ since $\pi_*(Z)$ is regular on W. Therefore, the flow-bow theorem can be applied (Godbillon considers that Z is complete, but the completeness does not intervene in any manner in this situation, all the more so as the Ehresmann's theorem we want here to prove is THE theorem (!) which allows precisely to know if a vector field is complete or not as we will show just after the present proof). Applying the flow-box theorem, we obtain, on the one hand, that:

- The flow $\varphi(f,t)$ is of class C^r $(r \geqslant 1)$ like Z (Godbillon considers rather that Z is of class C^r and φ of class C^{r-1} with respect to f, but we can show [Arn74, pp. 55–56, pp. 219–225] that it is not the case and that φ is of the same class C^r; and this is why Godbillon must take $r \geqslant 2$; this is not necessary and we only need to have $r \geqslant 1$).

- There exists a diffeomorphism on $U \cap \pi^{-1}(W)$ of class C^r $(r \geqslant 1)$ such that Z is diffeomorphic to $\widetilde{Z}$ which is constant on $U \cap \pi^{-1}(W)$. So, there is [Arn74, p. 56] a system of local coordinates $(y_1, \ldots, y_{k+m})$ on $U \cap \pi^{-1}(W)$ where $k + m = \dim E$ and $(\tilde{z}_1, \ldots, \tilde{z}_m)$ on W such that

$$\widetilde{Z} \equiv \frac{\partial}{\partial y_1} \qquad \text{on } U \cap \pi^{-1}(W)$$

and

$$\pi_*(\widetilde{Z}) \equiv \frac{\partial}{\partial \tilde{z}_1} \qquad \text{on } W.$$

- Thus, we can define the flow of $\widetilde{Z}$ we denote by $\tilde{\varphi} : (f,t) \in U \cap \pi^{-1}(W) \times I \longrightarrow E$ where I is a connected, compact interval containing $[0,1]$. This flow $\tilde{\varphi}$ trivializes locally E and this constitutes a first prerequisite to have a (differential) fiber bundle structure. From then on, we can also define $\tilde{h} : f \in U \cap \pi^{-1}(W) \longrightarrow E$ such that $\tilde{h}(f) \equiv \tilde{\varphi}(f,1) \equiv f_1$. And thus, we induce a map $\tilde{\mu}$ such that $\pi \circ \tilde{h} \equiv \tilde{\mu} \circ \pi$ and $f_1 \in F_{\tilde{\mu}(x)}$ where $x \in \pi(f)$ and $\tilde{\mu}(x) \equiv x_1 \in \pi(f_1)$. In addition, $\tilde{h}(U \cap \pi^{-1}(W))$ is a relatively compact open as well as $\tilde{\mu}(W)$.

From the flow definition, we have necessarily $\tilde{h}(U \cap \pi^{-1}(W)) \subseteq U \cap \pi^{-1}(W)$.

$*$ But, what still needs to be established is that if the fibers F_x $(x \in W)$ are non-connected, then these fibers are 1) matched biunivocally with the map $\tilde{h}$ (in other words, we must prove that $\tilde{h}$ is an isomorphism of cohomology groups $H^0(F_x)$ whenever $x \in W$), and 2) there is only a finite number of such fibers, *i.e.*, the Betti number b_0 is finite. This involves that taken a finite number N_x of *unspecified* connected components of F_x, then $\tilde{h}$ maps the union $\cup_{i=1}^{N_x} C_i$ to the union $\cup_{i=1}^{N_x} \tilde{h}(C_i) \subset F_{\tilde{\mu}(x)}$ which is also constituted of N_x connected components $\widetilde{C}_k$. If this situation is not encountered, it is then that the number of connected components is less than $\widetilde{N}_x$, *i.e.*, $\widetilde{N}_x < N_x$. Indeed, because $\tilde{h}$ is continuous, it maps any connected set to an image connected set, but these image connected sets are not always connected two-by-two even if their corresponding source sets are. Nevertheless, $\tilde{h}$ is invertible and its inverse is continuous from the definition of the flow. Hence, necessarily, we have $\widetilde{N}_x = N_x$ and this proves the first point.

$*$ We start with the assumption that $N_x = +\infty$ for all $x \in W$. And then, we consider two points $x_1 \equiv \pi(f_1) \in W$ and $x_2 \equiv \pi(\tilde{h}(f_1)) = \pi(f_2) = \tilde{\mu}(x_1) \in W$ but with $\tilde{h}(f_1) \neq f_2$ because $\tilde{h}(f_1)$ is not the connected component of f_2. Beside, x_1 and x_2 are in the same arc-connected component in W since x_2 is an image of the flow with the initial condition x_1. Thus, we assume that W is arc-connected (and thus, in particular, connected). It follows that whenever M is an union W of arc-connected opens, then M is arc-connected. But, any connected, topological (differential) manifold is arc-connected because it is homeomorphic to the arc-connected set $\mathbb{R}^m$; from which the required hypothesis of the theorem is deduced.

Furthermore, the series $\{x_i\}_{i=1,\ldots,+\infty}$ where $x_{i+1} = \tilde{\mu}(x_i)$ has an adherent point in $\overline{W}$ because the open set $W \subset V$ is relatively compact, whereas we could take a series $\{f_i\}_{i=1,\ldots,+\infty}$ of $f_i \in \pi^{-1}(x_i)$ with no adherent point and arranging (since $N_x = +\infty$) to have always $f_{i+1} \notin \tilde{h}(C_i)$ where C_i is the connected component containing f_i. But this would be a contradiction because we would obtain that a series with an adherent point in a compact $\overline{W}$ would have a corresponding inverse image series with no adherent point in $\pi^{-1}(\overline{W})$. This is not possible because π is proper and thus $\pi^{-1}(\overline{W})$ is compact. And as a result, the series has necessarily an adherent point. Hence, the fiber homotopy is invariant and we obtain a standard fiber and a differential fiber bundle structure. □

Godbillon gives some precisions if we consider manifolds with boudaries:

- If E and M have boundaries, their boundaries are put in correspondence with the map π which remains a surmersion;
- If E only has boundaries, then π remains a surmersion on the boundaries of E.

The second part of this proof induces the following corollary:

> "**Corollary:** *With the same hypotheses and notations than those used in the previous Ehresmann's theorem on completeness, we consider that E is a differential fiber bundle with standard fiber F. Then F is compact and it has a finite number of connected components, i.e.,* $\dim H^0(F) = b_0 < +\infty$*, where b_0 is the Betti number of order* 0."

We add the word '*completeness*' to designate this theorem due to Ehresmann because it generalizes, actually, the theorems [Ave83] of '*completeness*' of the vector fields [Ave83, pp. 78–79, and Corollary 4.6, p.79] which indicate that the existence of '*first integrals*' defining *compact* manifolds ensures the completeness of the fields.

Or also, whatever is any given compact K and any solution $f(t)$ of a given ODE defined by a complete vector field, then, the set of values t such that $f(t) \in K$ is also a compact (interval) in $\mathbb{R}$. In other words, if the vector field defining the ODE is not complete, we can find non-compact sets of values for t giving compact sets of values for $f(t)$ (ex.: $f(t) = 1/(t-1)$). This means also that the map f is not '*proper*', *i.e.*, a map is proper if the inverse image of a compact is a compact (we recall the the image of a compact is a compact if f is continuous).

Furthermore, from these theorems, we just need to have '*locally Lipschitzian*' vector fields on Banach spaces. Locally Lipschitzian means also continuous though not necessarily differentiable. This condition of Lipschitzian locality is thus weaker than the differentiability condition needed in the Ehresmann's theorem. But, nevertheless, the flow-box theorem invoked in the theorems of Frobenius and Ehresmann is based on the use of the Picard map which is shown to be contracting because it is locally Lipschitzian [Arn74]. Actually, any morphism of topological manifolds is locally Lipschitzian because it is defined on a Banach space $\mathbb{R}^n$. Hence, we have a transport of the Banach space structure on the topological manifolds onto which the flow-box theorem can be applied. Nevertheless, the Ehresmann's theorem of completeness needs the existence of differentiable maps or vector fields. Actually, we just need K-Lipschitzian vector fields to obtain flows of class C^1 [Ave83, p.75, Theorem 3.4, and p.76, Theorem 3.10].

In his 1947 paper published in the CRAS, Ehresmann gives a second important proposition from the notion of '*contact element*' of dimension $n = \dim M$, *i.e.*, fields of hyperplans of dimension n in E. Denoting by k de dimension of the standard fiber F, *i.e.*, $k = \dim F$, then, a field of hyperplans of dimension n is defined by k linear equations with coefficients in the presheaf $\mathcal{O}_E$, or, equivalently, it is defined by k differential exterior 1-forms $\omega_j \in T^*E$ ($j = 1, \ldots, k$) constituting a Pfaff system $S \equiv \{\omega_1, \ldots, \omega_k\}$. Moreover, Ehresmann considers these hyperplans to be '*secant*' to the fibers F_x ($x \in M$). This is another way of saying that the vectors contained in these hyperplanes are transversal to the vectors tangent to the fibers F_x. Moreover, these vectors are transversal only if there are *nonvanishing*. But then, each 1-form in S is transversal to the whole set of particular 1-forms defined as the pull-backs by $\pi : E \longrightarrow M$ of 1-forms defined on M. But still says otherwise, we can set:

Definition. *Let $U \subset M$ be any open in M, $\pi^{-1}(U) \subset E$ the corresponding open in E and $\psi : \pi^{-1}(U) \longrightarrow U \times F$ a local, differential trivialization defined on $\pi^{-1}(U)$. Then, the Pfaff system S on E such that $|S| = \dim F$ is said to be* "transversal" *to M if on any open U then* each *1-form $\omega \in S$ is the pull-back by $p_2 \circ \psi$ of a 1-form $\theta \in T^*F$ everywhere nonvanishing on F ($\equiv p_2 \circ \psi(\pi^{-1}(U))$), i.e., $p_2 \circ \psi^*(\theta) \equiv \omega$, where $p_2 : M \times F \longrightarrow F$.*

In this second proposition, Ehresmann assumes this field of contact elements to be secant and completely integrable. This means that S is transversal to M and completely integrable on E, and thus, that S verifies the Frobenius conditions. In addition, Ehresmann specifies that F is compact, but this property of compactness is induced if we assume the same hypotheses than those given in his first proposition (*i.e.*, the previous Ehresmann's theorem of completeness). Thus, the integral manifolds defined by S are coverings of M, and Ehresmann considers in particular the universal covering $\widehat{M}$ of M, and then the trivial fiber bundle $\hat{\pi} : \widehat{M} \times F \longrightarrow \widehat{M}$. Then, the second Ehresmann proposition is as follows:

"**Ehresmann's theorem on coverings.** *We set again the same notations and hypotheses than those used in the Ehresmann's theorem on completeness. Then, let S be a Pfaff system transversal to M and completely integrable on the differential bundle E such that $|S| = \dim F$, where F is the standard fiber of E. Denoting by $\widehat{M}$ the universal covering of M and considering the second projection $\hat{\pi} : \widehat{M} \times F \longrightarrow \widehat{M}$,* *then, there exists a map r' such that the diagram*

$$\begin{array}{ccc} \widehat{M}\times F & \xrightarrow{r'} & E \\ {\scriptstyle\hat{\pi}}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ \widehat{M} & \xrightarrow{r} & M \end{array}$$

is commutative, i.e., $\pi \circ r' = r \circ \hat{\pi}$, and, in addition, such that this map r'

1. *defines $\widehat{M}\times F$ as a covering of E, and*
2. *sends each set $\widehat{M}\times\{f\}$, where $f \in F$, to an integral manifold $\mathcal{I}_e$ of S in E passing through $e \in E$."*

Thus, the property *2.* in this theorem involves that each integral manifold $\mathcal{I}_e$ of S passing through $e \in E$ defines a covering of M, and consequently, that the surmersion π is a local homeomorphism from $\mathcal{I}_e$ to M. Therefore, for any point $m \in M$, the inverse image of m by π in $\mathcal{I}_e$, *i.e.*, $\pi^{-1}(m) \cap \mathcal{I}_e$, is a finite set of points in E of which the cardinal is less than the finite number of connected components of the standard fiber F (equal to $b_0 = \dim H^0(F)$). Hence, $\pi : \mathcal{I}_e \longrightarrow M$ is a covering of M with q sheets ($q \leqslant b_0 \leqslant +\infty$).

Moreover, if M is compact, then, necessarily, its fundamental (Poincaré) group $\pi_1(M)$ is finite, and thus, each integral manifold of S in E is compact. Indeed, it follows that $\widehat{M}$ is compact, and then, $\widehat{M}\times F$ is compact also since F is compact.

In addition, M being connected, if we assume also that M is 'simply connected,' *i.e.*, $\pi_1(M) = \{1\}$, then we deduce that E is isomorphic to $M\times F$.

But in all cases, and with the hypotheses of the previous theorem, we obtain also (again with the same notations and hypotheses than those used in the Ehresmann's theorem on completeness):

"**Corollary.** *The homotopy groups of E are isomorphic to those of $M\times F$:*

$$\pi_k(E) \simeq \pi_k(M\times F)\,.\text{''}$$

Lastly, we have the third flagship Ehresmann's theorem which is the very expression of the Weyl's gauge of length, called also Weyl's length of connection, and which generalizes the notions of connection due to É. Cartan (again with the same notations and…):

"**Ehresmann's theorem on connections.** *Let S be a Pfaff system* "transversal" *to M defined on the differential fiber bundle E and such that $|S| = \dim F$, where F is the standard fiber of E. Then,*

- *to each differential path γ linking x to x' in M corresponds a well-determined* homeomorphism *H_γ from F_x to $F_{x'}$,*
- *to the set $\pi_1(M, x)$ of loops $\ell \subset M$ with origin x corresponds a group $Aut(F_x)$ of automorphisms of F_x which can be considered as the structural Lie group G of the fiber bundle E, and moreover,*
- *if S is* 'completely integrable' *on E, then we obtain a representation (i.e., a monomorphism, thus non-surjective) of the fundamental group $\pi_1(M, x)$ in G.*"

Compared to the projective Cartan connections, the situation is a little "baroque," because F being compact then this standard fiber can only be the projective space $P\mathbb{R}^k \simeq F$ itself, and thus, E should be $\mathbb{R}^{k+1}$ and $M \simeq \mathbb{R}^*$. But then, it follows that M is non-connected. Therefore, E must be an union of Poincaré semi-spaces[52] of decreasing dimensions beginning with the dimension $k+1$: we obtain then the connected space $E \simeq \cup_{j=1}^{k+1} H^j$, and after only we obtain that $M \simeq \mathbb{R}^{*+}$ is connected.

Hence, $\pi : E \longrightarrow M$ defines the affine spaces of dimensions k in E which are, on the one hand, hyperplans in E or, on the other hand, those parts in E of the sphere S^k which are homeomorphic to open (Poincaré) disks of dimensions k.

Also, because 1) $\pi_1(\mathbb{R}^{*,+}) = \{1\}$, and 2) the structural Lie group is not trivial, then, no completely integrable and transversal Pfaff system S on M can be used to represent $\pi_1(\mathbb{R}^{*,+})$ better than the identity in the structural Lie group of E. Then, the Pfaff system S is defined, somehow, on a open disk of dimension k, and then, there exist such completely integrable and

[52] Each of these semi-spaces must be a connected open manifold defined by a local, always positive coordinate on $\mathbb{R}^n$; And thus without boundaries, but such that its closure is a connected boundary, *i.e.*, for instance, the closure of H^{k+1} has a unique connected boundary $\mathbb{R}^k$.

transversal systems. Hence, the Ehresmann 1-forms of connection (which define differential projective spaces) can be locally everywhere completely integrable and transversal; and thus there are also globally defined.

Appendix H: Proof of the Theorem 1

We recall that we must mainly prove that $\mathcal{EI}_o^+$ is a fiber bundle over $\mathcal{I}_o^+$.

Proof. Let K be a compact set such that $K \subset \mathcal{I}_o^+$. We denote by $\mathcal{P}_K^+ \subset \mathcal{P}_o^+$ the set of future-directed timelike paths $\gamma_{o,e}$ where $e \in K$: $\mathcal{P}_K^+ \equiv \{\gamma_{o,e}$ timelike; $e \in K \subset \mathcal{I}_o^+\}$. Moreover, because $\mathcal{M}$ is time oriented, we can also provide $\{o\} \cup \mathcal{I}_o^+$ with an elliptic metric h defined from the Lorentzian metric g and the time-like vector field ξ time orienting the spacetime manifold $\mathcal{M}$ [Ste51, p.206][Ave63, §2, pp.108–112]. We assume also that every set $\{o\} \cup \mathcal{I}_o^+$ is t-complete [Ave63, Definition (3, II), p.140] (or, equivalently, *complete* in the sense of Ehresmann [Ehr51, §8, p.50][Ehr50, §6, p.166]),[53] *i.e.*, any path $\gamma_{o,e} \in \mathcal{P}_o^+$ are the image sets of maps (curves) of class C^1: $\iota : [0,1] \longrightarrow \mathcal{I}_o^+$, such that $\iota(0) = o$ and $\iota(1) = e$. Equivalently, any path $\gamma_{o,e} \subset \{o\} \cup \mathcal{I}_o^+$ is homeomorphic to a timelike path $\widehat{\gamma_{o,e}} \subset T_a\mathcal{M}$ for all $a \in \gamma_{o,e}$ where $T_a\mathcal{M}$ is the tangent Minkowski spacetime at the event a. Then, because K is compact and $\{o\} \cup \mathcal{I}_o^+$ is t-complete, any path $\gamma_{o,e} \in \mathcal{P}_K^+$ is contained in a subset $\mathcal{E}_K^+ \subset \{o\} \cup \mathcal{I}_o^+$ in one-to-one correspondence with a subset $\widehat{\mathcal{E}_K^+}$ of $T_o\mathcal{M}$ which is bounded for the Euclidean distance defined by h at o and contained in the closure of the future cone of $T_o\mathcal{M}$ for the metric g at o.

Furthermore, we denote by $\mathcal{C}_K^+$ the set of events which are elements of a path in $\mathcal{P}_K^+$. The former is the union of compact sets contained in the same bounded set $\mathcal{E}_K^+$, and thus, $\mathcal{C}_K^+$ is compact. Additionally, $\mathcal{C}_K^+$ is in a one-to-one correspondence with a compact set $\widehat{\mathcal{C}_K^+} \subset T_o\mathcal{M}$ which is the set of points which are elements of a path $\widehat{\gamma_{o,e}} \subset T_o\mathcal{M}$. Moreover, we provide $\widehat{\mathcal{C}_K^+}$ with the topology $\widehat{\mathcal{T}^0}$ of the uniform convergence defined from the Euclidean distance defined by the elliptic metric h at the event o. Then, $\widehat{\mathcal{T}^0}$ provides the topology $\mathcal{T}^0$ of the uniform convergence on $\mathcal{C}_K^+$. Then, because the latter is a compact metric space, it is, equivalently, a

[53] The 1950 version is the incomplete version of the 1951's one.

complete metric space for the uniform convergence [Dix81, Theorem 5.6.1, p.76]. Therefore, the set of maps from the compact $[0,1]$ to the compact metric space $\mathcal{C}_K^+$ is complete for the uniform convergence [Dix81, Theorem 6.1.6, p.80]. In other words, $\mathcal{P}_K^+$ is complete for the uniform convergence, and thus, it is closed. Additionally, because $\mathcal{P}_K^+$ is also bounded, it is compact.

Then, let $\textstyle\int\alpha$ be the map such that

$$\textstyle\int\alpha : \gamma_{o,e} \in \mathcal{P}_o^+ \longrightarrow \epsilon = \displaystyle\int_{\gamma_{o,e}} \alpha \in \mathbb{R}\,,$$

and e and $\tilde{e}$ two events in K. We denote by $D(\gamma_{o,e},\, \gamma_{o,\tilde{e}})$ and $d(\gamma_{o,e},\, \gamma_{o,\tilde{e}})$ the values such that

$$D(\gamma_{o,e},\, \gamma_{o,\tilde{e}}) \equiv \left| \int_{\gamma_{o,e}} \alpha - \int_{\gamma_{o,\tilde{e}}} \alpha \right| = \left| \int_0^1 (\iota^*(\alpha) - \tilde{\iota}^*(\alpha)) \right|,$$

and

$$d(\gamma_{o,e},\, \gamma_{o,\tilde{e}}) \equiv \sup_{t\in[0,\,1]} \|\iota(t) - \tilde{\iota}(t)\|\,,$$

where $\iota([0,\,1]) = \gamma_{o,e}$ and $\tilde{\iota}([0,\,1]) = \gamma_{o,\tilde{e}}$. Then, α is Lipschitzian on $\mathcal{P}_K^+$ because it is of class C^0 on the compact set $\mathcal{P}_K^+$. As a result, we obtain that there exists a constant $c_K(\alpha)$ depending on α and K such that $D(\gamma_{o,e},\, \gamma_{o,\tilde{e}}) \leqslant c_K(\alpha)\, d(\gamma_{o,e},\, \gamma_{o,\tilde{e}})$ from which we deduce that $\textstyle\int\alpha$ is continuous on $\mathcal{P}_K^+$. Hence, we deduce that $(\textstyle\int\alpha)\,(\mathcal{P}_K^+) \subset \mathbb{R}$ is compact [Dix81, Theorem 4.2.12, p.54]. Then, the map Γ_o is a *proper* differentiable surmersion and because $\mathcal{P}_K^+$ is connected, we deduce from Ehresmann's theorem [Ehr47a, Proposition 1, p.1611][Ehr51, Proposition, p.31] that 1) the fibration $\Gamma_o : \mathcal{E}\mathcal{I}_o^+ \longrightarrow \mathcal{I}_o^+$ is locally trivializable, *i.e.*, $\mathcal{E}\mathcal{I}_o^+$ is a fiber bundle, and 2) either $\mathcal{E}\mathcal{I}_o^+$ or all the preimage $\Gamma_o^{-1}(e)$ are compact sets (note that if $\mathcal{I}_o^+$ is the homotopy class $[\gamma_{o,e}]$ of a path $\gamma_{o,e}$ we can apply Avez's lemma [Ave63, Lemme $(1, \mathrm{II})$, p.141]). $\square$

Appendix I: The inverse maps

We have the relations:

$$\begin{cases} \rho_1 = (\tau_1)^2 + (\tau_2)^2 + (\tau_3)^2 + (\tau_4)^2 \\ \rho_2 = 2\,(\tau_1\tau_2 + \tau_3\tau_4) \\ \rho_3 = 2\,(\tau_1\tau_3 + \tau_2\tau_4) \\ \rho_4 = 2\,(\tau_1\tau_4 + \tau_3\tau_2) \end{cases} \tag{I1}$$

and we want to have the coordinates τ_i with respect to the coordinates κ_j, *i.e.*, we must find the inverse maps. First, we show that $a \equiv \sum_{\alpha=1}^{4} \tau_\alpha > 0$ is expressed with the κ^α's only. Indeed, we have: $a^2 = \sum_{\alpha=1}^{4} \rho_\alpha$. Hence, because $a > 0$, we deduce that

$$\sum_{\alpha=1}^{4} \tau_\alpha = \sqrt{\textstyle\sum_{\beta=1}^{4} \rho_\beta}\,. \tag{I2}$$

Then, we have the following system of equations:

$$\begin{cases} (\tau_1 + \tau_3)\,(\tau_2 + \tau_4) = \dfrac{1}{2}\,(\rho_2 + \rho_4)\,, \\ (\tau_1 + \tau_3) + (\tau_2 + \tau_4) = \sqrt{\textstyle\sum_{\alpha=1}^{4} \rho_\alpha}\,, \end{cases} \tag{I3}$$

from which we obtain for one of the inverse map ($\epsilon_a = \pm 1$):

$$\begin{cases} \tau_1 + \tau_3 = \dfrac{1}{2}\,\left(\sqrt{\rho_1 + \rho_3 + \rho_2 + \rho_4} + \epsilon_a \sqrt{\rho_1 + \rho_3 - \rho_2 - \rho_4}\,\right), \\ \tau_2 + \tau_4 = \dfrac{1}{2}\,\left(\sqrt{\rho_1 + \rho_3 + \rho_2 + \rho_4} - \epsilon_a \sqrt{\rho_1 + \rho_3 - \rho_2 - \rho_4}\,\right). \end{cases} \tag{I4}$$

But, we have also the two supplementary systems of equations:

$$\begin{cases} (\tau_1 + \tau_2)\,(\tau_3 + \tau_4) = \dfrac{1}{2}\,(\rho_3 + \rho_4)\,, \\ (\tau_1 + \tau_2) + (\tau_3 + \tau_4) = \sqrt{\textstyle\sum_{\alpha=1}^{4} \rho_\alpha}\,, \end{cases} \qquad \begin{cases} (\tau_1 + \tau_4)\,(\tau_3 + \tau_2) = \dfrac{1}{2}\,(\rho_2 + \rho_3)\,, \\ (\tau_1 + \tau_4) + (\tau_3 + \tau_2) = \sqrt{\textstyle\sum_{\alpha=1}^{4} \rho_\alpha}\,, \end{cases} \tag{I5}$$

and then, for one of the inverse map, we deduce that

$$\begin{cases} \tau_1 + \tau_2 = \dfrac{1}{2}\,\left(\sqrt{\rho_1 + \rho_3 + \rho_2 + \rho_4} + \epsilon_b \sqrt{\rho_1 + \rho_2 - \rho_3 - \rho_4}\,\right), \\ \tau_3 + \tau_4 = \dfrac{1}{2}\,\left(\sqrt{\rho_1 + \rho_3 + \rho_2 + \rho_4} - \epsilon_b \sqrt{\rho_1 + \rho_2 - \rho_3 - \rho_4}\,\right), \\ \tau_1 + \tau_4 = \dfrac{1}{2}\,\left(\sqrt{\rho_1 + \rho_3 + \rho_2 + \rho_4} + \epsilon_c \sqrt{\rho_1 + \rho_4 - \rho_2 - \rho_3}\,\right), \\ \tau_3 + \tau_2 = \dfrac{1}{2}\,\left(\sqrt{\rho_1 + \rho_3 + \rho_2 + \rho_4} - \epsilon_c \sqrt{\rho_1 + \rho_4 - \rho_2 - \rho_3}\,\right). \end{cases} \tag{I6}$$

where $\epsilon_{b,c} = \pm 1$. Thus, from the relations (I4) and (I6), we obtain the formulas:

$$\begin{aligned} \tau_1 = \frac{1}{4}\,\Big(&\sqrt{\rho_1 + \rho_2 + \rho_3 + \rho_4} + \epsilon_a \sqrt{\rho_1 + \rho_3 - \rho_2 - \rho_4} \\ &+ \epsilon_b\,\sqrt{\rho_1 + \rho_2 - \rho_3 - \rho_4} + \epsilon_c \sqrt{\rho_1 + \rho_4 - \rho_2 - \rho_3}\;\Big)\,, \end{aligned} \tag{I7a}$$

$$\begin{aligned} \tau_2 = \frac{1}{4}\,\Big(&\sqrt{\rho_1 + \rho_2 + \rho_3 + \rho_4} - \epsilon_a \sqrt{\rho_1 + \rho_3 - \rho_2 - \rho_4} \\ &+ \epsilon_b\,\sqrt{\rho_1 + \rho_2 - \rho_3 - \rho_4} - \epsilon_c \sqrt{\rho_1 + \rho_4 - \rho_2 - \rho_3}\;\Big)\,, \end{aligned} \tag{I7b}$$

$$\tau_3 = \frac{1}{4}\,\big(\sqrt{\rho_1+\rho_2+\rho_3+\rho_4} + \epsilon_a\sqrt{\rho_1+\rho_3-\rho_2-\rho_4} \\ - \epsilon_b\,\sqrt{\rho_1+\rho_2-\rho_3-\rho_4} - \epsilon_c\sqrt{\rho_1+\rho_4-\rho_2-\rho_3}\;\big)\,, \tag{I7c}$$

$$\tau_4 = \frac{1}{4}\,\big(\sqrt{\rho_1+\rho_2+\rho_3+\rho_4} - \epsilon_a\sqrt{\rho_1+\rho_3-\rho_2-\rho_4} \\ - \epsilon_b\,\sqrt{\rho_1+\rho_2-\rho_3-\rho_4} + \epsilon_c\sqrt{\rho_1+\rho_4-\rho_2-\rho_3}\;\big)\,. \tag{I7d}$$

Therefore, there are exactly eight inverse maps corresponding to the eight strata of the generalized Tzitzeica surface $\mathcal{T}i^3$.

We deduce also the $\tau_{(\kappa)\alpha}$'s such that $\tau_\alpha = e^{\frac{1}{2}\kappa^1}\tau_{(\kappa)\alpha}$. For instance, we have:

$$\tau_{(\kappa)1} = \frac{1}{4}\,\Big(\sqrt{1+\kappa^2+\kappa^3+\kappa^4} + \epsilon_a\sqrt{1+\kappa^3-\kappa^2-\kappa^4} \\ + \epsilon_b\,\sqrt{1+\kappa^2-\kappa^3-\kappa^4} + \epsilon_c\sqrt{1+\kappa^4-\kappa^2-\kappa^3}\,.\Big)\,, \tag{I8}$$

Appendix J: The traces of the Riemann tensors

The following computations is based on the results and the notations used by R.S. Kulkarni [Kul70].

First, note that Kulkarni defines the Riemann tensor by the following formula (see footnote 1, p.313 in [Kul70]). Let X, Y and Z be three vector fields then the Riemann tensor R Kulkarni defines is the following

$$R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z\,.$$

Thus, it is the opposite of the current definition we use from now and throughout this appendix and in the whole of this document:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\,. \tag{J1}$$

then, we consider two metrics $\bar{g}$ and g such that

$$\bar{g} = e^{2\varphi}\,g\,. \tag{J2}$$

Then, denoting by $\overline{R}$ the Riemann tensor associated with $\bar{g}$ and R the one associated with g, we have the formula [Kul70, p.318] (note that in this formula the sign differs from the one in the Kulkarni formula because of the different definition we take for the Riemann tensor):

$$\overline{R}(X,Y)Z = R(X,Y)Z - T(X,Y)Z\,, \tag{J3}$$

where

$$\begin{aligned} T(X,Y)Z = &\Big(Q(Y,Z) + g(Y,Z)\,\|G\|^2\Big)\,X - \Big(Q(X,Z) + g(X,Z)\,\|G\|^2\Big)\,Y \\ &+ g(Y,Z)\,Q_0(X) - g(X,Z)\,Q_0(Y)\,, \end{aligned} \tag{J4}$$

where

$$\|G\|^2 \equiv g(G,G)\,, \tag{J5a}$$

$$G = grad(\varphi)\,, \tag{J5b}$$

$$g(X,G) \equiv i_X d\varphi\,, \tag{J5c}$$

$$Q(X,Y) = hess_\varphi(X,Y) - d\varphi(X)\,d\varphi(Y)\,, \tag{J5d}$$

$$hess_\varphi(X,Y) = i_X d(d\varphi(Y)) - d\varphi(\nabla_X Y) = g(Y, \nabla_X G)\,, \tag{J5e}$$

$$Q_0(X) = \nabla_X G - d\varphi(X)\,G\,. \tag{J5f}$$

Then, let $\{Z_1, \ldots, Z_n\}$ be a basis of vector fields of the tangent bundle and $\{Z^{*1}, \ldots, Z^{*n}\}$ its dual cobasis, then the trace of the tensor T is such that

$$Tr(\overline{R}(X,Y)) = Tr(R(X,Y)) - Tr(T(X,Y))\,, \tag{J6}$$

where

$$Tr(T(X,Y)) = \sum_{i=1}^{n} Z^{*i}(T(X,Y)\,Z_i) \tag{J7}$$

$$\begin{aligned} = \sum_{i=1}^{n} Z^{*i}\,\Big\{&\Big(Q(Y,Z_i) + g(Y,Z_i)\,\|G\|^2\Big)\,X - \Big(Q(X,Z_i) + g(X,Z_i)\,\|G\|^2\Big)\,Y \\ &+ g(Y,Z_i)\,Q_0(X) - g(X,Z_i)\,Q_0(Y)\Big\} \end{aligned} \tag{J8}$$

In particular, we have

$$\begin{aligned}\sum_{i=1}^{n} Z^{*i}(g(Y,Z_i)\,Q_0(X)) &= \sum_{i=1}^{n} g(Y,Z_i)\,Z^{*i}(Q_0(X))\\ &= \sum_{i=1}^{n} g(Y,Z_i\,Z^{*i}(Q_0(X)))\\ &= g(Y,\sum_{i=1}^{n} Z_i\,Z^{*i}(Q_0(X)))\\ &= g(Y,Q_0(X))\,. \end{aligned}\tag{J9}$$

Also, we obtain

$$\begin{aligned}\sum_{i=1}^{n} Z^{*i}\left(\left(Q(Y,Z_i)+g(Y,Z_i)\,\|G\|^2\right)X\right) &= \sum_{i=1}^{n}\left(Q(Y,Z_i)+g(Y,Z_i)\,\|G\|^2\right)Z^{*i}(X)\\ &= Q(Y,\sum_{i=1}^{n} Z_i\,Z^{*i}(X))+g(Y,\sum_{i=1}^{n} Z_i\,Z^{*i}(X))\,\|G\|^2\\ &= Q(Y,X)+g(Y,X)\,\|G\|^2\,. \end{aligned}\tag{J10}$$

Therefore, the trace of $T(X,Y)$ is such that

$$Tr(T(X,Y)) = Q(Y,X)+g(Y,X)\,\|G\|^2 - Q(X,Y) - g(X,Y)\,\|G\|^2 \tag{J11}$$

$$\qquad + g(Y,Q_0(X)) - g(X,Q_0(Y))$$

$$= g(Y,Q_0(X)) - g(X,Q_0(Y))\,. \tag{J12}$$

Besides, from the definition of Q_0, we deduce that

$$\begin{aligned}g(Y,Q_0(X)) &= g(Y,\nabla_X G) - d\varphi(X)\,g(Y,G)\\ &= g(Y,\nabla_X G) - d\varphi(X)\,d\varphi(Y)\end{aligned}\tag{J13}$$

and thus, we obtain

$$Tr(T(X,Y)) = g(Y,\nabla_X G) - g(X,\nabla_Y G)\,. \tag{J14}$$

Moreover, from the property of the covariant derivative, we have

$$i_X d(d\varphi(Y)) = i_X d(g(Y,G)) = g(\nabla_X Y,G) + g(Y,\nabla_X G) = d\varphi(\nabla_X Y) + g(Y,\nabla_X G)\,, \tag{J15}$$

and therefore, we have also

$$g(Y, \nabla_X G) = i_X d(g(Y, G)) - d\varphi(\nabla_X Y) = i_X d(i_Y d\varphi) - d\varphi(\nabla_X Y)\,. \tag{J16}$$

Then, we obtain

$$\begin{aligned} Tr(T(X,Y)) &= i_X d(i_Y d\varphi) - i_Y d(i_X d\varphi) + d\varphi(\nabla_Y X - \nabla_X Y) \\ &= d(d\varphi)(X,Y) + d\varphi([X,Y]) + d\varphi(\nabla_Y X - \nabla_X Y) \\ &= d\varphi([X,Y] + \nabla_Y X - \nabla_X Y) \\ &= -d\varphi(Tor(X,Y))\,, \end{aligned} \tag{J17}$$

where Tor is the torsion tensor of the metric g. Finally, we obtain that

$$Tr(\overline{R}(X,Y)) = Tr(R(X,Y)) + d\varphi(Tor(X,Y))\,. \tag{J18}$$

Hence, the traces of the two Riemann tensors are equal if and only if the metric connection defined from ∇ (or $\overline{\nabla}$) is torsion-free.

Also, we have the fundamental relation between the scalar curvatures $\overline{Sc}$ and Sc [Kul70] (with a difference of sign because of the different convention in the definition of the Riemann tensor):

$$\overline{Sc} = e^{-2\varphi}\left(Sc + n(n-1)\|G\|^2 + 2(n-1)\,Tr(Q_0)\right). \tag{J19}$$

Appendix K: The Christoffel symbols

Computing the Christoffel symbols $\Gamma^{\beta}_{\alpha\mu}$ of the metric g given by (2.1) in the base $\{\partial_{\tau_1}, \ldots, \partial_{\tau_4}\}$ and its dual base $\{d\tau_1, \ldots, d\tau_4\}$, we obtain the following expressions:

- if $\alpha \neq \beta$, $\mu \neq \beta$, $\alpha \neq \mu$ and where $\zeta \neq \alpha$, μ, β:

$$\begin{aligned} \Gamma^{\alpha}_{\mu\beta} = \frac{1}{6\,\nu_\alpha}\Bigg\{ &\frac{2}{\nu_\alpha}\Big(\nu_\mu\left(\partial_\alpha\nu_\beta - \partial_\beta\nu_\alpha\right) + \nu_\beta\left(\partial_\alpha\nu_\mu - \partial_\mu\nu_\alpha\right)\Big) - \left(\partial_\mu\nu_\beta + \partial_\beta\nu_\mu\right) \\ &\frac{1}{\nu_\zeta}\Big(\nu_\beta\left(\partial_\mu\nu_\zeta - \partial_\zeta\nu_\mu\right) + \nu_\mu\left(\partial_\beta\nu_\zeta - \partial_\zeta\nu_\beta\right)\Big)\Bigg\}\,, \end{aligned} \tag{K1a}$$

- if $\alpha \neq \beta$:

$$\Gamma^{\alpha}_{\beta\alpha} = \frac{\nu_\beta}{6}\left\{\frac{1}{2}\left(\frac{1}{\nu_\beta}\partial_\beta + \frac{1}{\nu_\alpha}\partial_\alpha\right)\ln(|g|) + \frac{2}{\nu_\beta\nu_\alpha}\left(\partial_\beta\nu_\alpha - \partial_\alpha\nu_\beta\right) - \left(\sum_{\mu=1}^{4}\frac{1}{\nu_\mu}\partial_\mu\right)\ln(\nu_\beta\,\nu_\alpha)\right\}, \tag{K1b}$$

- if $\alpha \neq \beta$:

$$\Gamma^{\alpha}_{\beta\beta} = \frac{\nu_\beta}{3}\,\partial_\beta \ln\left(\frac{|g|^{1/2}}{\nu_\beta(\nu_\alpha)^2}\right), \tag{K1c}$$

- and for any α:

$$\Gamma^{\alpha}_{\alpha\alpha} = \frac{2}{3}\,\partial_\alpha \ln\left(\nu_\alpha\,|g|^{1/4}\right). \tag{K1d}$$

Moreover, we define the Pfaffian system of 1-forms σ_i defined by (2.8), *i.e.*, $\sigma_\alpha \equiv \nu_\alpha\, d\tau_\alpha$, and such that $g \equiv -\sum_{\alpha<\beta=1}^{4}\sigma_\alpha \odot \sigma_\beta$. Then, because the dimension of $\mathcal{M}$ is the number of such 1-forms σ_β, the Pfaffian system $Pf \equiv \{\sigma_1, \ldots, \sigma_4\}$ is necessarily completely integrable locally on $\mathcal{M}$. Therefore, there exist structure functions $\mathcal{C}^{\alpha\beta}_{\mu}$ such that $\mathcal{C}^{\alpha\beta}_{\mu} = -\mathcal{C}^{\beta\alpha}_{\mu}$ and

$$d\sigma_\mu = \sum_{\alpha,\beta=1}^{4}\mathcal{C}^{\alpha\beta}_{\mu}\,\sigma_\alpha \wedge \sigma_\beta\,. \tag{K2}$$

Then, the functions ν_α must satisfy the following system of relations and partial differential equations :

$$\mathcal{C}^{\beta\mu}_{\alpha} = 0\,, \qquad \alpha \neq \beta,\ \beta \neq \mu,\ \alpha \neq \mu\,, \tag{K3a}$$

$$\partial_\alpha\nu_\beta = \mathcal{C}_{\alpha,\beta}\,\nu_\alpha\,\nu_\beta\,, \qquad \alpha \neq \beta\,, \tag{K3b}$$

where $\mathcal{C}_{\alpha,\beta} \equiv 2\,\mathcal{C}^{\alpha\beta}_{\beta}$, and then, $\mathcal{C}_{\alpha,\alpha} = 0$. If we denote respectively by $\mathcal{C}_{[\alpha\beta]}$ and $\mathcal{C}_{(\alpha\beta)}$ the antisymmetric and the symmetric parts of the twelve nonvanishing structure functions $\mathcal{C}_{\alpha,\beta}$, then, we have the following results for the 1-forms $\Gamma^{\alpha}_{\beta} \equiv \sum_{\mu=1}^{4}\Gamma^{\alpha}_{\beta\mu}\,d\tau_\mu$:

- if $\alpha = \beta$:

$$\Gamma^{\alpha}_{\alpha} = \frac{1}{\nu_\alpha^2}(\partial_{\tau_\alpha}\nu_\alpha)\,\sigma_\alpha + \frac{1}{3}\left\{\left(\sum_{\beta=1}^{4}\mathcal{C}_{\alpha,\beta}\right)\sigma_\alpha + \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{4}\left(2\,\mathcal{C}_{(\alpha,\beta)} + \sum_{\gamma=1}^{4}\left(\mathcal{C}_{[\alpha\gamma]} + \mathcal{C}_{[\beta\gamma]}\right)\right)\sigma_\beta\right\}, \tag{K4a}$$

- if $\alpha \neq \beta$:

$$\Gamma^{\alpha}_{\beta} = \frac{\nu_\alpha}{3\,\nu_\beta}\left\{\left(\sum_{\gamma=1}^{4}\mathcal{C}_{\alpha,\gamma} - 3\,\mathcal{C}_{\alpha,\beta}\right)\sigma_\alpha + \left(2\,\mathcal{C}_{(\alpha\beta)} + \sum_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{4}\left(\mathcal{C}_{[\alpha\gamma]} + \mathcal{C}_{[\beta\gamma]}\right)\right)\sigma_\beta \right.$$
$$\left. + \sum_{\substack{\gamma=1\\ \gamma\neq\alpha,\beta}}^{4}\left(2\left(\mathcal{C}_{[\beta\alpha]} + \mathcal{C}_{[\beta\gamma]}\right) - \mathcal{C}_{(\alpha\gamma)} + \sum_{\substack{\chi=1\\ \chi\neq\alpha,\beta,\gamma}}^{4}\left(\mathcal{C}_{[\alpha\chi]} + \mathcal{C}_{[\gamma\chi]}\right)\right)\sigma_\gamma\right\}, \quad \text{(K4b)}$$

And, additionally, we have also:

$$Tr(\Gamma) = \sum_{\alpha=1}^{4}\left(\frac{1}{\nu_\alpha^2}\left(\partial_{\tau_\alpha}\nu_\alpha\right) + \sum_{\beta=1}^{4}\mathcal{C}_{\alpha,\beta}\right)\sigma_\alpha\,. \quad \text{(K5)}$$

Also, from the definition (4.56a) of G and the relation (4.57), we have

$$\left(\mathcal{V}^{-1}d\mathcal{V} - \mathcal{V}^{-1}\Gamma\mathcal{V}\right)\widehat{\mathfrak{H}} + \widehat{\mathfrak{H}}^{\,t}\left(\mathcal{V}^{-1}d\mathcal{V} - \mathcal{V}^{-1}\Gamma\mathcal{V}\right) = 0\,, \quad \text{(K6)}$$

and therefore, $\mathcal{V}^{-1}d\mathcal{V} - \mathcal{V}^{-1}\Gamma\mathcal{V}$ satisfies the same relations (4.61) than σ, *i.e.*, we have the relations:

$$\sum_{\beta=1,\,\beta\neq\alpha}^{4}\frac{1}{\nu_\beta}\,\Gamma^{\alpha}_{\beta} = 0\,, \quad \text{(K7a)}$$

$$\frac{\nu_\alpha}{\nu_\beta}\,\Gamma^{\alpha}_{\beta} + \frac{\nu_\beta}{\nu_\alpha}\,\Gamma^{\beta}_{\alpha} - \Gamma^{\alpha}_{\alpha} - \Gamma^{\beta}_{\beta} + d\ln(\nu_\alpha\,\nu_\beta) = 0\,, \quad \text{(K7b)}$$

$$\sum_{\beta=1,\,\beta\neq\alpha}^{4}\nu_\beta\,\Gamma^{\beta}_{\alpha} = 2\left(\nu_\alpha\,\Gamma^{\alpha}_{\alpha} - d\nu_\alpha\right)\,. \quad \text{(K7c)}$$

Appendix L: The infinitesimal automorphisms of the Yano-Ihsihara projecting 1-form $\hat{\pi}$

We want to compute the Lie algebra $\mathcal{L}(\hat{\pi})$ of vector fields $\xi \in \chi(\widehat{\mathcal{M}})$ such that $\mathcal{L}_\xi\hat{\pi} = \lambda\,\hat{\pi}$ for $\lambda \in \mathcal{O}_{\widehat{\mathcal{M}}}$ and where

$$\hat{\pi} = d\tau_5 + e^{-\tau_5}\textstyle\sum_{\alpha=1}^{4} e^{\tau_\alpha}\,d\tau_\alpha\,.$$

For, we use an Hamiltonian characterization of $\mathcal{L}(\hat{\pi})$ that can be the following. Indeed, $\hat{\pi}$ can be considered also as a Darboux 1-form. Hence, let H be the contact Hamiltonian (moment map) defined by $H : \xi \in \mathcal{L}(\hat{\pi}) \longrightarrow i_\xi\hat{\pi} = f \in \mathcal{O}_{\widehat{\mathcal{M}}}$. The interest of such map H is in the computations of the vector $\xi \in \mathcal{L}(\hat{\pi})$. Indeed, we have $\mathcal{L}_\xi\hat{\pi} = i_\xi d\hat{\pi} + d(i_\xi\hat{\pi}) = \lambda\,\hat{\pi}$, and then, we deduce first from $i_\xi\hat{\pi} = f$ that

$$\xi^5 = f - e^{-\tau_5}\Big(\textstyle\sum_{\alpha=1}^4 e^{\tau_\alpha}\,\xi^\alpha\Big) \tag{L1}$$

and, second, from $d\hat{\pi} = \hat{\pi}\wedge d\tau_5$, that we have also the relation:

$$i_\xi d\hat{\pi} = f\,d\tau_5 - \xi^5\,\hat{\pi}\,. \tag{L2}$$

Therefore, $\mathcal{L}\,\hat{\pi}$ is defined by

$$\mathcal{L}\,\hat{\pi} = \left(\partial_5 f + e^{-\tau_5}\sum_{\alpha=1}^4 e^{\tau_\alpha}\,\xi^\alpha\right)\hat{\pi} + \sum_{\alpha=1}^4\Big(\partial_\alpha f - e^{\tau_\alpha-\tau_5}\,(f+\partial_5 f)\Big)\,d\tau_\alpha\,. \tag{L3}$$

Thus, the function f must satisfy the following set of equations for each α:

$$\partial_5 f - e^{\tau_5-\tau_\alpha}\,\partial_\alpha f + f = 0\,. \tag{L4}$$

The general solution for f is then such that

$$f = e^{-\tau_5}\,F(\varsigma^0)\,, \tag{L5}$$

where $\varsigma^0 = \sum_{\mathfrak{k}=1}^5 e^{\tau_\mathfrak{k}}$ and $F \in \mathcal{O}_\mathbb{R}$. As a result, the general expression for $\xi \in \mathcal{L}(\hat{\pi})$ is the following:

$$\xi = e^{-\tau_5}\left(F(\varsigma^0) - \sum_{\alpha=1}^4 e^{\tau_\alpha}\,\xi^\alpha\right)\partial_5 + \sum_{\alpha=1}^4 \xi^\alpha\,\partial_\alpha\,. \tag{L6}$$

In particular, we see that H is bijective since f defines univocally the vector field ξ. We conclude that $\mathcal{L}(\hat{\pi})$ is generated by the five following vector fields on $\widehat{\mathcal{M}}$:

$$\eta_5 = e^{-\tau_5}\,\partial_5\,, \tag{L7a}$$

$$\eta_\alpha = \partial_\alpha - e^{\tau_\alpha-\tau_5}\,\partial_5\,. \tag{L7b}$$

on the ring of real functions (invariants) $F(\varsigma^0)$. But now, considering the new variables $\varsigma^\mathfrak{k} = e^{\tau_\mathfrak{k}}$, then, we have that

$$\frac{\partial}{\partial\varsigma^\mathfrak{k}} = e^{-\tau_\mathfrak{k}}\,\frac{\partial}{\partial\tau_\mathfrak{k}}\,. \tag{L8}$$

Therefore, we can rewrite the vectors in the following forms:

$$\zeta_5 = \frac{\partial}{\partial \varsigma^5}\,, \tag{L9a}$$

$$\zeta_\alpha = \varsigma^\alpha \left(\frac{\partial}{\partial \varsigma^\alpha} - \frac{\partial}{\partial \varsigma^5}\right). \tag{L9b}$$

Then, we see that $\mathcal{L}(\hat{\pi})$ is a solvable Lie algebra. Additionally, any Hamiltonian $\mathcal{H}$ is depending on the "time" variable τ_5, and the variables τ_α with their conjugate moments ς^α and, moreover, it must satisfy the following Hamilton equations:

$$\dot{\tau}_\alpha = \frac{\partial \mathcal{H}}{\partial \varsigma^\alpha} \tag{L10a}$$

$$\dot{\varsigma}^\alpha \equiv \varsigma^\alpha\, \dot{\tau}_\alpha = -\frac{\partial \mathcal{H}}{\partial \tau_\alpha} \tag{L10b}$$

$$\dot{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial \tau_5} \tag{L10c}$$

where the dot indicates the total derivatives with respect to τ_5 .

Appendix M: The Christoffel symbols of $\hat{g}$ in $\widehat{\mathcal{M}}$

We apply the relations (9.28), *i.e.*,

$$\widehat{\Gamma}^{\mathfrak{u}}_{\mathfrak{s},\mathfrak{h}} = \frac{1}{2}\sum_{\mathfrak{v}=1}^{5} \hat{g}^{\mathfrak{u}\mathfrak{v}}\Big\{\hat{g}(\hat{\xi}_\mathfrak{v}, [\hat{\xi}_\mathfrak{h}, \hat{\xi}_\mathfrak{s}]) + \hat{g}(\hat{\xi}_\mathfrak{h}, [\hat{\xi}_\mathfrak{v}, \hat{\xi}_\mathfrak{s}]) + \hat{g}(\hat{\xi}_\mathfrak{s}, [\hat{\xi}_\mathfrak{v}, \hat{\xi}_\mathfrak{h}]) + \hat{\xi}_\mathfrak{h}(\hat{g}_{\mathfrak{v}\mathfrak{s}}) + \hat{\xi}_\mathfrak{s}(\hat{g}_{\mathfrak{v}\mathfrak{h}}) - \hat{\xi}_\mathfrak{v}(\hat{g}_{\mathfrak{s}\mathfrak{h}})\Big\}, \tag{M1}$$

with the relations of commutation (9.26), *i.e.*,

$$[\hat{\xi}_\alpha, \hat{\xi}_\beta] = \hat{\xi}_\alpha - \hat{\xi}_\beta\,, \qquad [\hat{\xi}_5, \hat{\xi}_\alpha] = \hat{\xi}_\alpha + \hat{\xi}_5\,. \tag{M2}$$

First of all, it easy to see the first important relation:

$$\widehat{\Gamma}^5_5 = 0\,, \tag{M3}$$

and then we compute the terms $\widehat{\Gamma}^5_{\alpha,\mathfrak{k}}$. We have:

$$\widehat{\Gamma}^5_{\alpha,\mathfrak{h}} = \frac{1}{2}\Big\{\hat{g}(\hat{\xi}_5, [\hat{\xi}_\mathfrak{h}, \hat{\xi}_\alpha]) + \hat{g}(\hat{\xi}_\mathfrak{h}, [\hat{\xi}_5, \hat{\xi}_\alpha]) + \hat{g}(\hat{\xi}_\alpha, [\hat{\xi}_5, \hat{\xi}_\mathfrak{h}]) + \hat{\xi}_\mathfrak{h}(\hat{g}_{5\alpha}) + \hat{\xi}_\alpha(\hat{g}_{5\mathfrak{h}}) - \hat{\xi}_5(\hat{g}_{\alpha\mathfrak{h}})\Big\}. \tag{M4}$$

Hence, if $\mathfrak{h} = 5$, we obtain:

$$\begin{aligned}
\widehat{\Gamma}^5_{\alpha,5} &= \frac{1}{2}\Big\{\hat{g}(\hat{\xi}_5, [\hat{\xi}_5, \hat{\xi}_\alpha]) + \hat{g}(\hat{\xi}_5, [\hat{\xi}_5, \hat{\xi}_\alpha]) + \hat{g}(\hat{\xi}_\alpha, [\hat{\xi}_5, \hat{\xi}_5]) + \hat{\xi}_5(\hat{g}_{5\alpha}) + \hat{\xi}_\alpha(\hat{g}_{55}) - \hat{\xi}_5(\hat{g}_{\alpha 5})\Big\} \\
&= \hat{g}(\hat{\xi}_5, [\hat{\xi}_5, \hat{\xi}_\alpha]) = \hat{g}(\hat{\xi}_5, \hat{\xi}_\alpha + \hat{\xi}_5) \\
&= 1\,, \qquad \text{(M5)}
\end{aligned}$$

and for $\mathfrak{h} = \beta$:

$$\begin{aligned}
\widehat{\Gamma}^5_{\alpha,\beta} &= \frac{1}{2}\Big\{\hat{g}(\hat{\xi}_5, [\hat{\xi}_\beta, \hat{\xi}_\alpha]) + \hat{g}(\hat{\xi}_\beta, [\hat{\xi}_5, \hat{\xi}_\alpha]) + \hat{g}(\hat{\xi}_\alpha, [\hat{\xi}_5, \hat{\xi}_\beta]) + \hat{\xi}_\beta(\hat{g}_{5\alpha}) + \hat{\xi}_\alpha(\hat{g}_{5\beta}) - \hat{\xi}_5(\hat{g}_{\alpha\beta})\Big\} \\
&= \frac{1}{2}\Big\{\hat{g}(\hat{\xi}_5, \hat{\xi}_\beta - \hat{\xi}_\alpha) + \hat{g}(\hat{\xi}_\beta, \hat{\xi}_5 + \hat{\xi}_\alpha) + \hat{g}(\hat{\xi}_\alpha, \hat{\xi}_5 + \hat{\xi}_\beta) - \hat{\xi}_5(\hat{g}_{\alpha\beta})\Big\} \\
&= \hat{g}_{\alpha\beta} - \frac{1}{2}\,\hat{\xi}_5(\hat{g}_{\alpha\beta}) \qquad \text{(M6)} \\
&= \widehat{\Gamma}^5_{\beta,\alpha}
\end{aligned}$$

Now, we compute the terms $\widehat{\Gamma}^\alpha_{\mathfrak{u},\mathfrak{k}}$ and we have first the terms $\widehat{\Gamma}^\alpha_{5,\mathfrak{h}}$:

$$\widehat{\Gamma}^\alpha_{5,\mathfrak{h}} = \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\Big\{\hat{g}(\hat{\xi}_\beta, [\hat{\xi}_\mathfrak{h}, \hat{\xi}_5]) + \hat{g}(\hat{\xi}_\mathfrak{h}, [\hat{\xi}_\beta, \hat{\xi}_5]) + \hat{g}(\hat{\xi}_5, [\hat{\xi}_\beta, \hat{\xi}_\mathfrak{h}]) + \hat{\xi}_5(\hat{g}_{\beta\mathfrak{h}})\Big\}. \qquad \text{(M7)}$$

Thus, if $\mathfrak{h} = 5$, we obtain:

$$\begin{aligned}
\widehat{\Gamma}^\alpha_{5,5} &= \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\Big\{\hat{g}(\hat{\xi}_\beta, [\hat{\xi}_5, \hat{\xi}_5]) + \hat{g}(\hat{\xi}_5, [\hat{\xi}_\beta, \hat{\xi}_5]) + \hat{g}(\hat{\xi}_5, [\hat{\xi}_\beta, \hat{\xi}_5]) + \hat{\xi}_5(\hat{g}_{\beta 5})\Big\} \\
&= \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\Big\{\hat{g}(\hat{\xi}_5, [\hat{\xi}_\beta, \hat{\xi}_5]) + \hat{g}(\hat{\xi}_5, [\hat{\xi}_\beta, \hat{\xi}_5])\Big\} \\
&= \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\Big\{-2\hat{g}(\hat{\xi}_5, \hat{\xi}_\beta + \hat{\xi}_5)\Big\} \\
&= -\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\, q_\beta\,, \qquad \text{(M8)} \\
&= -\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\, \widehat{\Gamma}^5_{\beta,5}\,, \qquad \text{(M9)}
\end{aligned}$$

where $q_\alpha = 1$ for all $\alpha = 1, \ldots, 4$. And, if $\mathfrak{h} = \gamma$:

$$\begin{aligned}
\widehat{\Gamma}^{\alpha}_{5,\gamma} &= \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\Big\{\hat{g}(\hat{\xi}_\beta, [\hat{\xi}_\gamma, \hat{\xi}_5]) + \hat{g}(\hat{\xi}_\gamma, [\hat{\xi}_\beta, \hat{\xi}_5]) + \hat{g}(\hat{\xi}_5, [\hat{\xi}_\beta, \hat{\xi}_\gamma]) + \hat{\xi}_5(\hat{g}_{\beta\gamma})\Big\} \\
&= \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\Big\{\hat{g}(\hat{\xi}_\beta, [\hat{\xi}_\gamma, \hat{\xi}_5]) + \hat{g}(\hat{\xi}_\gamma, [\hat{\xi}_\beta, \hat{\xi}_5]) + \hat{\xi}_5(\hat{g}_{\beta\gamma})\Big\} \\
&= \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\Big\{-2\hat{g}_{\beta\gamma} + \hat{\xi}_5(\hat{g}_{\beta\gamma})\Big\} = \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\,\hat{\xi}_5(\hat{g}_{\beta\gamma}) - \sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\hat{g}_{\beta\gamma} \\
&= \frac{1}{2}\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\,\hat{\xi}_5(\hat{g}_{\beta\gamma}) - \delta^{\alpha}_{\gamma} && \text{(M10)} \\
&= -\sum_{\mu=1}^{4} \hat{g}^{\alpha\mu}\,\widehat{\Gamma}^{5}_{\mu,\gamma}\,. && \text{(M11)}
\end{aligned}$$

In particular, we deduce from (M9) and (M11) that

$$\widehat{\Gamma}^{\alpha}_{5} = -\sum_{\beta=1}^{4} \hat{g}^{\alpha\beta}\,\widehat{\Gamma}^{5}_{\beta}\,. \tag{M12}$$

Now, it remains compute the terms $\widehat{\Gamma}^{\alpha}_{\beta,\mathfrak{k}}$ and we have first the terms $\widehat{\Gamma}^{\alpha}_{\beta,5}$:

$$\begin{aligned}
\widehat{\Gamma}^{\alpha}_{\beta,5} &= \frac{1}{2}\sum_{\gamma=1}^{4} \hat{g}^{\alpha\gamma}\Big\{\hat{g}(\hat{\xi}_\gamma, [\hat{\xi}_5, \hat{\xi}_\beta]) + \hat{g}(\hat{\xi}_5, [\hat{\xi}_\gamma, \hat{\xi}_\beta]) + \hat{g}(\hat{\xi}_\beta, [\hat{\xi}_\gamma, \hat{\xi}_5]) + \hat{\xi}_5(\hat{g}_{\gamma\beta}) \\
&\qquad + \hat{\xi}_\beta(\hat{g}_{\gamma 5}) - \hat{\xi}_\gamma(\hat{g}_{\beta 5})\Big\} \\
&= \frac{1}{2}\sum_{\gamma=1}^{4} \hat{g}^{\alpha\gamma}\Big\{\hat{g}(\hat{\xi}_\gamma, [\hat{\xi}_5, \hat{\xi}_\beta]) + \hat{g}(\hat{\xi}_\beta, [\hat{\xi}_\gamma, \hat{\xi}_5]) + \hat{\xi}_5(\hat{g}_{\gamma\beta})\Big\} \\
&= \frac{1}{2}\sum_{\gamma=1}^{4} \hat{g}^{\alpha\gamma}\,\hat{\xi}_5(\hat{g}_{\gamma\beta})\,. && \text{(M13)}
\end{aligned}$$

In particular, we deduce the important relation:

$$\widehat{\Gamma}^{\alpha}_{[\beta,5]} \equiv \widehat{\Gamma}^{\alpha}_{\beta,5} - \widehat{\Gamma}^{\alpha}_{5,\beta} = \delta^{\alpha}_{\beta}\,. \tag{M14}$$

Let us notice again that we have no symmetric Christoffel symbols although the torsion is vanishing, *i.e.*, we have $\widehat{\Gamma}^{\mathfrak{r}}_{\mathfrak{u},\mathfrak{v}} \neq \widehat{\Gamma}^{\mathfrak{r}}_{\mathfrak{v},\mathfrak{u}}$ in general. This is due to the fact that the basis vectors $\hat{\xi}^{\mathfrak{u}}$

are non-commuting vector fields (see Exemple 2, p.97). And, lastly, the terms $\widehat{\Gamma}^{\alpha}_{\beta,\gamma}$:

$$\begin{aligned}
\widehat{\Gamma}^{\alpha}_{\beta,\gamma} &= \frac{1}{2}\sum_{\mu=1}^{4}\hat{g}^{\alpha\mu}\Big\{\hat{g}(\hat{\xi}_\mu,[\hat{\xi}_\gamma,\hat{\xi}_\beta]) + \hat{g}(\hat{\xi}_\gamma,[\hat{\xi}_\mu,\hat{\xi}_\beta]) + \hat{g}(\hat{\xi}_\beta,[\hat{\xi}_\mu,\hat{\xi}_\gamma]) + \hat{\xi}_\gamma(\hat{g}_{\mu\beta}) \\
&\qquad + \hat{\xi}_\beta(\hat{g}_{\mu\gamma}) - \hat{\xi}_\mu(\hat{g}_{\beta\gamma})\Big\} \\
&= \frac{1}{2}\sum_{\mu=1}^{4}\hat{g}^{\alpha\mu}\Big\{\hat{g}(\hat{\xi}_\mu,\hat{\xi}_\gamma-\hat{\xi}_\beta) + \hat{g}(\hat{\xi}_\gamma,\hat{\xi}_\mu-\hat{\xi}_\beta) + \hat{g}(\hat{\xi}_\beta,\hat{\xi}_\mu-\hat{\xi}_\gamma) + \hat{\xi}_\gamma(\hat{g}_{\mu\beta}) \\
&\qquad + \hat{\xi}_\beta(\hat{g}_{\mu\gamma}) - \hat{\xi}_\mu(\hat{g}_{\beta\gamma})\Big\} \\
&= \frac{1}{2}\sum_{\mu=1}^{4}\hat{g}^{\alpha\mu}\Big\{2\hat{g}_{\mu\gamma} - 2\hat{g}_{\beta\gamma} + \hat{\xi}_\gamma(\hat{g}_{\mu\beta}) + \hat{\xi}_\beta(\hat{g}_{\mu\gamma}) - \hat{\xi}_\mu(\hat{g}_{\beta\gamma})\Big\} \\
&= q_\beta\,\delta^\alpha_\gamma - \hat{g}_{\beta\gamma}\left\{\sum_{\mu=1}^{4}\hat{g}^{\alpha\mu}\,q_\mu\right\} + \frac{1}{2}\sum_{\mu=1}^{4}\hat{g}^{\alpha\mu}\Big\{\hat{\xi}_\gamma(\hat{g}_{\mu\beta}) + \hat{\xi}_\beta(\hat{g}_{\mu\gamma}) - \hat{\xi}_\mu(\hat{g}_{\beta\gamma})\Big\}. \qquad \text{(M15)}
\end{aligned}$$

In particular, we have

$$\widehat{\Gamma}^{\alpha}_{[\alpha,\beta]} = \widehat{\Gamma}^{\alpha}_{\alpha,\beta} - \widehat{\Gamma}^{\alpha}_{\beta,\alpha} = \delta^\alpha_\beta - 1 \qquad \text{(M16)}$$

and

$$\sum_{\alpha=1}^{4}\left(\widehat{\Gamma}^{\alpha}_{\beta,\alpha} - \widehat{\Gamma}^{\alpha}_{\alpha,\beta}\right) = 3\,. \qquad \text{(M17)}$$

But also:

$$\widehat{\Gamma}^{\alpha}_{[\beta,\gamma]} = \widehat{\Gamma}^{\alpha}_{\beta,\gamma} - \widehat{\Gamma}^{\alpha}_{\gamma,\beta} = q_\beta\,\delta^\alpha_\gamma - q_\gamma\,\delta^\alpha_\beta\,, \qquad \text{(M18)}$$

Lastly, we deduce that

$$\begin{aligned}
Tr(\widehat{\Gamma}) &\equiv \sum_{\mathfrak{k},\mathfrak{h}=1}^{5}\widehat{\Gamma}^{\mathfrak{k}}_{\mathfrak{k},\mathfrak{h}}\,\hat{\pi}^{\mathfrak{h}} = \sum_{\alpha=1}^{4}\widehat{\Gamma}^{\alpha}_{\alpha} = \frac{1}{2}\sum_{\mathfrak{k}=1}^{5}\left(\sum_{\alpha,\beta=1}^{4}\hat{g}^{\alpha\beta}\,\hat{\xi}_{\mathfrak{k}}(\hat{g}_{\alpha\beta})\right)\hat{\pi}^{\mathfrak{k}} \\
&= \frac{1}{2}\,d\left(\ln(|\hat{g}|)\right), \qquad \text{(M19)}
\end{aligned}$$

where we denote by $|\hat{g}|$ the absolute value of the determinant $\det(\hat{g})$ of $\hat{g}$ in the coframe $\{d\varsigma^1,\ldots,d\varsigma^5\}$.

In addition, we have the remarkable relation in the coframe $\{d\varsigma^1,\ldots,d\varsigma^5\}$:

$$\det(\hat{g}) = e^{-2\left(4\varphi+\sum_{\mathfrak{k}=1}^{5}\tau_{\mathfrak{k}}\right)}\,\det(g)\,, \qquad \text{(M20)}$$

where

$$\det(g) = -\frac{3}{16}\left(\prod_{\alpha=1}^{4}\hat{\nu}_\alpha\right)^2 , \tag{M21}$$

is the determinant of g in the coframe $\{d\tau_1, \ldots, d\tau_4\}$.

In the coframe $\widehat{\mathcal{B}}^* \equiv \{\hat{\pi}^1, \ldots, \hat{\pi}^5\}$, the determinant of $\hat{g}$ is the following:

$$\det \hat{g} = e^{-2\left(4\varphi - 4\,\tau_5 + \sum_{\alpha=1}^{4}\tau_\alpha\right)}\,\det(g)\,, \tag{M22}$$

where $\det g$ is again the determinant of g in the coframe $\{d\tau_1, \ldots, d\tau_4\}$.

Appendix N: The projective geodesics on $\widehat{\mathcal{M}}$

Projective geodesics generated by a projective connection are defined as follows (there are geodesics on $\mathcal{M}$ and $\widehat{\mathcal{M}}$). First, each 5-vector field $\hat{u} \in \chi(\widehat{\mathcal{M}})$ defines a horizontal 4-vector field u such that $u \equiv \hat{u} - u^5\,\mathring{\xi}_5$. Second, a projective line is, actually, a two dimensional plane in the tangent space $T_\varsigma\widehat{\mathcal{M}}$ at $\varsigma \in \widehat{\mathcal{M}}$ (intersecting possibly the affine projective hyperplane such that $u^5 = 1$; and similar to $\varsigma^0 = 1$ for the homogeneous projective coordinates). Hence, a projective line $\mathscr{L}_\varsigma$ in $T_\varsigma\widehat{\mathcal{M}}$ is the vector space of vectors $\hat{u}$ generated by two vectors $\hat{v}$ and $\hat{v}'$ linearly independent, and intersecting the affine projective hyperplane (possibly at infinity) such that $u^5 = 1$. Therefore, each 5-vector $\hat{u}$ of a given projective line can be written as the following linear combination:

$$\hat{u} \equiv \alpha\,\hat{v} + \alpha'\,\hat{v}'\,, \tag{N1}$$

where α and α' are smooth functions on $\widehat{\mathcal{M}}$. Also, $\widehat{\nabla}_{\hat{u}}\hat{u}$ is an element of this given projective line $\mathscr{L}$ if there exist two other smooth functions θ and θ' on $T\widehat{\mathcal{M}}$ (and not only on $\widehat{\mathcal{M}}$) such that

$$\widehat{\nabla}_{\hat{u}}\hat{u} \equiv \theta\,\hat{v} + \theta'\,\hat{v}'\,. \tag{N2}$$

To highlight the relations between the vectors $\hat{v}$ and $\hat{v}'$ with the projective hyperplane defined by $u^5 = 1$, we decompose also these two vectors into a sum of a horizontal vector field $\hat{w}$ such that $\mathring{\pi}^5(\hat{w}) = 0$ and a multiple of $\mathring{\xi}_5$, forming both a new basis of the projective line $\mathscr{L}$. Thus, in full generality, we can again write the expressions (N1) and (N2) in the forms:

$$\hat{u} \equiv u^5 \, \mathring{\xi}_5 + \hat{w} \,, \qquad \widehat{\nabla}_{\hat{u}} \hat{u} \equiv \theta^5 \, \mathring{\xi}_5 + \theta \, \hat{w} \,, \tag{N3}$$

where $\hat{w} \neq 0$. Clearly, projecting on the horizontal subspace, this is equivalent to write:

$$\underline{\widehat{\nabla}_{\hat{u}} \hat{u}} \equiv \theta \, \hat{w} \,. \tag{N4}$$

And, in particular, if $\hat{u} \equiv u$, *i.e.*, if the projective line $\mathscr{L}$ is at infinity, then in this case we obtain that

$$\underline{\widehat{\nabla}_{u} u} \equiv \theta \, u \,. \tag{N5}$$

Finally, and most fundamentally, we can write:

$$\widehat{\nabla}_{\hat{u}} \hat{u} - \theta \, \hat{u} \equiv 0 \quad \mathrm{mod} \; (\mathring{\xi}_5) \,. \tag{N6}$$

Remark 19. *We must mention that the function θ can be also defined on $T\widehat{\mathcal{M}}$ and not only on $\widehat{\mathcal{M}}$, i.e., θ can depend on $\hat{u}$. This is an important feature to identify such factor θ in the sequel.*

Consequently, if $\hat{u} \equiv \sum_{\mathfrak{k}=1}^{4} u^{\mathfrak{k}} \, \mathring{\xi}_{\mathfrak{k}}$ then the precedent expression (N6) can be also written in the form ($\alpha = 1, \ldots, 4$):

$$i_{\hat{u}} du^5 + \sum_{\mathfrak{k}=1}^{5} u^{\mathfrak{k}} \, \mathring{\Gamma}^5_{\mathfrak{k}}(\hat{u}) = \theta \, u^5 + \theta^5 \,, \tag{N7a}$$

$$i_{\hat{u}} du^\alpha + \sum_{\mathfrak{k}=1}^{5} u^{\mathfrak{k}} \, \mathring{\Gamma}^\alpha_{\mathfrak{k}}(\hat{u}) = \theta \, u^\alpha \,, \tag{N7b}$$

where $i_{\hat{u}}$ is the interior product, and where we use indifferently the notation $i_{\hat{u}} \varphi$ or $\varphi(\hat{u})$ to indicate the contraction of a differential 1-form φ with a vector $\hat{u}$. The coordinate u^5 can be set equal, for instance, to a constant because the equation (N7a) is just a definition of the arbitrary function θ^5. More generally, u^5 can be equal to any function of $\varsigma \in \widehat{\mathcal{M}}$. Therefore, only the differential equations (N7b) are relevant. We see also that θ can be taken such that $\theta \equiv \theta' + \mathring{\Gamma}^5_5(\hat{u})$, and therefore, considering the projective Cartan connection ω such that $\omega \equiv \mathring{\Gamma} - \mathring{\Gamma}^5_5 \, \mathbb{1}$, the equations (N7b) can also be written as

$$i_{\hat{u}} du^\alpha + \sum_{\mathfrak{k}=1}^{5} u^{\mathfrak{k}} \, \omega^\alpha_{\mathfrak{k}}(\hat{u}) = \theta' \, u^\alpha \,. \tag{N8}$$

As a result, we obtain (taking care this time that we have: $i = 1, \ldots, 3$):

$$i_{\hat{u}} du^i + \sum_{\mathfrak{h}=1}^{5} \mathring{\Gamma}^i_{\mathfrak{h}}(\hat{u}) u^{\mathfrak{h}} = \frac{u^i}{u^4} \left(i_{\hat{u}} du^4 + \sum_{\mathfrak{h}=1}^{5} \mathring{\Gamma}^4_{\mathfrak{h}}(\hat{u}) u^{\mathfrak{h}} \right). \tag{N9}$$

In addition, we have also the relation:

$$i_{\hat{u}} du^i - \left(\frac{u^i}{u^4} \right) i_{\hat{u}} du^4 = u^4\, i_{\hat{u}} d \left(\frac{u^i}{u^4} \right),$$

and then, considering that the geodesic is parameterized by the parameter λ, we have:

$$i_{\hat{u}} d \equiv \frac{d}{d\lambda}. \tag{N10}$$

With this definition (N9) can be written as ($i = 1, \ldots, 3$):

$$u^4 \frac{d}{d\lambda} \left(\frac{u^i}{u^4} \right) = - \sum_{\mathfrak{h}=1}^{5} \mathring{\Gamma}^i_{\mathfrak{h}}(\hat{u})\, u^{\mathfrak{h}} + \frac{u^i}{u^4} \left(\sum_{\mathfrak{h}=1}^{5} \mathring{\Gamma}^4_{\mathfrak{h}}(\hat{u})\, u^{\mathfrak{h}} \right). \tag{N11}$$

Therefore, we can define the "3-vector" field $\vec{\zeta} = (\zeta^1, \zeta^2, \zeta^3)$ and ζ^4 and ζ^5 such that ($i = 1, \ldots, 3$):

$$\zeta^i \equiv \frac{u^i}{u^4}, \qquad \zeta^4 \equiv 1, \qquad \zeta^5 \equiv \frac{u^5}{u^4}, \tag{N12}$$

where u^5 is any function of $\varsigma \in \widehat{\mathcal{M}}$, and then, if (N11) is divided by $(u^4)^2 \neq 0$ (we take the square because we have terms such that $\mathring{\Gamma}^{\beta}_{\mathfrak{h}}(\hat{u})\, u^{\mathfrak{h}}$) we obtain ($i = 1, \ldots, 3$):

$$\frac{1}{u^4} \frac{d\zeta^i}{d\lambda} = - \sum_{\mathfrak{h}=1}^{5} \mathring{\Gamma}^i_{\mathfrak{h}}(\hat{\zeta})\, \zeta^{\mathfrak{h}} + \zeta^i \left(\sum_{\mathfrak{h}=1}^{5} \mathring{\Gamma}^4_{\mathfrak{h}}(\hat{\zeta})\, \zeta^{\mathfrak{h}} \right). \tag{N13}$$

where $\hat{\zeta} \equiv (\zeta^1, \zeta^2, \zeta^3, \zeta^4 \equiv 1, \zeta^5)$.

But, we can also rewritten the components $u^{\mathfrak{h}}$ of $\hat{u}$ in the form ($\mathfrak{h} = 1, \ldots, 5$):

$$u^{\mathfrak{h}} \equiv \frac{d\varsigma^{\mathfrak{h}}}{d\lambda}, \tag{N14}$$

and then, we obtain the following projective geodesic equations on $\widehat{\mathcal{M}}$ at $\varsigma \in \widehat{\mathcal{M}}$ ($i = 1, \ldots, 3$ and $\mathfrak{h} = 1, \ldots, 5$):

$$\zeta^{\mathfrak{h}} = \frac{d\varsigma^{\mathfrak{h}}}{d\varsigma^4}, \qquad \frac{d\zeta^i}{d\varsigma^4} = - \sum_{\mathfrak{u}=1}^{5} \mathring{\Gamma}^i_{\mathfrak{u}}(\hat{\zeta})\, \zeta^{\mathfrak{u}} + \zeta^i \left(\sum_{\mathfrak{v}=1}^{5} \mathring{\Gamma}^4_{\mathfrak{v}}(\hat{\zeta})\, \zeta^{\mathfrak{v}} \right). \tag{N15}$$

These equations can be written in the general form ($i = 1, \ldots, 3$):

$$\frac{d\zeta^i}{d\varsigma^4} = -P^i(\vec{\zeta}\,) + \zeta^i P^4(\vec{\zeta}\,)\,, \tag{N16}$$

where the $P^\alpha(\vec{\zeta}\,)$ ($\alpha = 1, \ldots, 4$) are non-homogeneous polynomials of degree 2 (in full generality) which depend on the 3 components of the 3-vector field $\vec{\zeta} \equiv (\zeta^1, \zeta^2, \zeta^3)$. The arbitrary component ζ^5 can be any function of the four coordinates ς^α ($\alpha = 1, \ldots, 4$).

The coefficients of these polynomials defined from the coefficients of $\mathring{\Gamma}$ can only depend on the coordinates ς^α ($\alpha = 1, \ldots, 4$) defining a local chart on $\mathcal{M}$, and thus, these projective geodesic equations *are non-holonomic equations because of their dependence with respect to* ς^4.

Beginning from equations of the form (N16), and considering they define geodesic equations obtained from a given projective connection $\mathring{\Gamma}$, the latter can be obtained from homogeneous quadratic polynomials Q^α defined themselves from polynomials P^α such that ($\alpha = 1, \ldots, 4$):

$$(u^4)^2\, P^\alpha(\vec{\zeta}\,) \equiv Q^\alpha(\hat{u}) \equiv \sum_{\mathfrak{h}=1}^{5} \mathring{\Gamma}^\alpha_{\mathfrak{h}}(\hat{u})\, u^{\mathfrak{h}}\,. \tag{N17}$$

But also, in full generality, because the geodesic equations are the same for connections of the form $\mathring{\Gamma}^\alpha_{\mathfrak{v}} + \delta^\alpha_{\mathfrak{v}}\,\psi$, we can set also ($\alpha = 1, \ldots, 4$ and $\mathfrak{v} = 1, \ldots, 5$):

$$Q^\alpha(\hat{u}) \equiv \sum_{\mathfrak{v}=1}^{5} (\mathring{\Gamma}^\alpha_{\mathfrak{v}}(\hat{u}) + \delta^\alpha_{\mathfrak{v}}\,\psi(\hat{u}))\, u^{\mathfrak{v}}\,. \tag{N18}$$

Or, equivalently, we have:

$$\sum_{\mathfrak{v}=1}^{5} \mathring{\Gamma}^\alpha_{\mathfrak{v}}(\hat{u})\, u^{\mathfrak{v}} = Q^\alpha(\hat{u}) - \psi(\hat{u})\, u^\alpha\,. \tag{N19}$$

Besides, the polynomials Q^i are homogeneous polynomials of degree 2, and thus, they verify the Euler identity:

$$\sum_{\mathfrak{v}=1}^{5} u^{\mathfrak{v}} \left(\frac{\partial Q^\alpha}{\partial u^{\mathfrak{v}}}\right) = 2\, Q^\alpha\,.$$

On another hand, the polynomials

$$R^\alpha(\hat{u}) = \sum_{\mathfrak{h}=1}^{5} \mathring{\Gamma}^\alpha_{\mathfrak{h}}(\hat{u})\, u^{\mathfrak{h}} \tag{N20}$$

are also homogeneous polynomials of degree 2 as the polynomial $u^\alpha\,\psi(u)$ which verifies a Euler identity which can be written in the form:

$$\sum_{\mathfrak{k}=1}^{5} u^{\mathfrak{k}}\left(\frac{\partial(\psi(\hat{u})\,u^\alpha)}{\partial u^{\mathfrak{k}}}\right) = 2\,u^\alpha\,\psi(\hat{u})\,.$$

Consequently, *if* $\mathring{\Gamma}$ *is "torsion-free,"* and only in this case, then the formula (N19) can be written as

$$2\,\mathring{\Gamma}^\alpha_{\mathfrak{k}}(\hat{u}) = \left(\frac{\partial Q^\alpha}{\partial u^{\mathfrak{k}}}\right) - \left(\frac{\partial(\psi\,u^\alpha)}{\partial u^{\mathfrak{k}}}\right). \tag{N21}$$

Indeed, the homogeneous polynomial $R^\alpha(\hat{u})$ defines always a unique symmetric bilinear form, and thus, a unique torsion-free connection. But, conversely, we have an infinite set of non-symmetric bilinear forms giving the same polynomial $R^\alpha(\hat{u})$, *i.e.*, an infinite set of connections with torsion.

Considering also that ψ is homogeneous of degree 1, *i.e.*, satisfying the relation:

$$\sum_{\mathfrak{k}=1}^{5} u^{\mathfrak{k}}\left(\frac{\partial\psi}{\partial u^{\mathfrak{k}}}\right) = \psi\,, \tag{N22}$$

we deduce that

$$\begin{aligned}\sum_{\mathfrak{k}=1}^{5}\frac{\partial Q^{\mathfrak{k}}}{\partial u^{\mathfrak{k}}} &= \sum_{\mathfrak{k}=1}^{5}\frac{\partial(\psi\,u^{\mathfrak{k}})}{\partial u^{\mathfrak{k}}} + 2\,Tr\,\mathring{\Gamma}\\ &= \sum_{\mathfrak{k}=1}^{5}\left(u^{\mathfrak{k}}\frac{\partial\psi}{\partial u^{\mathfrak{k}}} + \psi\right) + 2\,Tr\,\mathring{\Gamma}\\ &= \sum_{\mathfrak{k}=1}^{5} u^{\mathfrak{k}}\frac{\partial\psi}{\partial u^{\mathfrak{k}}} + 4\,\psi + 2\,Tr\,\mathring{\Gamma}\\ &= 5\,\psi + 2\,Tr\,\mathring{\Gamma}\,.\end{aligned}$$

Consequently, we have:

$$\psi = \frac{1}{5}\left(\sum_{\mathfrak{k}=1}^{5}\frac{\partial Q^{\mathfrak{k}}}{\partial u^{\mathfrak{k}}} - 2\,Tr\,\mathring{\Gamma}\right). \tag{N23}$$

Then, from (N21), we deduce finally:

$$2\left(\mathring{\Gamma}^\alpha_{\mathfrak{k}}(\hat{u}) - \frac{1}{5}\left(\frac{\partial(u^\alpha\,Tr\,\mathring{\Gamma})}{\partial u^{\mathfrak{k}}}\right)\right) = \frac{\partial Q^\alpha}{\partial u^{\mathfrak{k}}} - \frac{1}{5}\sum_{\mathfrak{v}=1}^{5}\left(\delta^\alpha_{\mathfrak{k}}\,\frac{\partial Q^{\mathfrak{v}}}{\partial u^{\mathfrak{v}}} + u^\alpha\,\frac{\partial^2\,Q^{\mathfrak{v}}}{\partial u^{\mathfrak{k}}\,\partial u^{\mathfrak{v}}}\right). \tag{N24}$$

Remark 20. *Therefore, only traceless projective connection* $\mathring{\Gamma}$ *is defined univocally from the polynomials* Q^α*. Nevertheless, we see that the equations* (N7b) *can be modified to obtain other equations of which the solutions are the same geodesics. Indeed, if we define* Γ *such that*

$$\Gamma^\alpha_{\mathfrak{k}}(\hat{u}) \equiv \mathring{\Gamma}^\alpha_{\mathfrak{k}}(\hat{u}) - \frac{1}{5}\,\frac{\partial(u^\alpha\, Tr\,\mathring{\Gamma})}{\partial u^{\mathfrak{k}}} \tag{N25}$$

then, we obtain

$$i_{\hat{u}} du^\alpha + \sum_{\mathfrak{k}=1}^{5} u^{\mathfrak{k}}\, \Gamma^\alpha_{\mathfrak{k}}(\hat{u}) = \tilde{\theta}(\hat{u})\, u^\alpha\,, \tag{N26}$$

where

$$\tilde{\theta}(\hat{u}) = \theta + \frac{1}{5}\left(i_{\hat{u}} d(Tr\,\mathring{\Gamma}) - Tr\,\mathring{\Gamma}\right). \tag{N27}$$

And thus, we can say also that Γ *is projectively equivalent to* $\mathring{\Gamma}$*, and then,* Γ *is completely defined from* (N24)*.*

Remark 21. *Hence, we see that these ordinary differential equations* (N16) *which are strongly nonlinear cubic in dimension* $n = 3$*, can be reduced to geodesic equations if they ar ewritten in a space of dimension greater than* $n + 1$ *(in this case, everything happens as if we had the equations* (N16) *in dimension* $n + 1$ *with the polynomial* $P^{n+1} = 0$*).*

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